On the well-posedness and the numerical approximation of some problems arising from the modelisation of metamaterials

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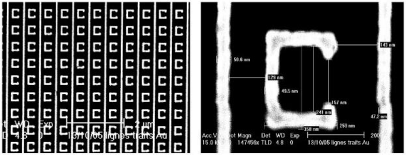
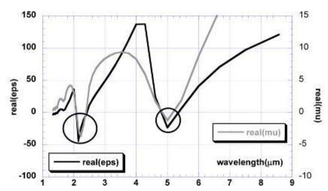
A definition...

Framework :

- Artificial medium created by homogenization of small physical components.
- Exotic behavior at some frequencies.



(a) Periodic array of SRR



(b) Real part of homogenized parameters

Fig.: Real part of ε and μ of a periodic array of SRR,
Kanté B., A. De Lustrac et als, Metamaterials for optical and radio communications.

Some application of metamaterials :

- Super-Lens (with materials with negative refractive index).
- Alice's mirror,
- Control of light (with photonic crystals).
- Invisibility, cloaking
- Seismic protection for buildings.
- ...

Setting of the problem

Hypothesis :

- $\Omega \subset \mathbb{R}^3$ = a bounded open set of $\mathbb{R}^3$ with $\mathcal{C}^2$ boundary with outward unitary normal denoted by $\mathbf{n}$.
- Metamaterials = compact material.

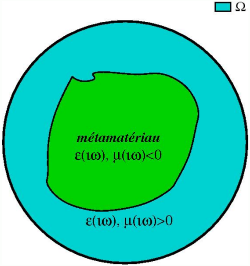


Fig.: Géometric setting of the problem

Maxwell's equation

Maxwell's system in Laplace transform :

$$\left\{ \begin{array}{l} \textit{Find } (E,H) \in \mathcal{D}(\mathbb{M}) \textit{ such that :} \\ (K(p,x) + \mathbb{M}) \begin{pmatrix} E \\ H \end{pmatrix} = f, \;\; L^2(\Omega)^6 \end{array} \right.$$

where :

$$\mathcal{D}(\mathbb{M}) = \left\{ (E,H) \in H(curl,\Omega)^2 \mid \mathbf{n}(x) \times (E + \Lambda(x)(\mathbf{n}(x) \times H)) = 0, \;\; H^{-1/2}(\partial\Omega) \right\}$$

$$p = iw + \eta, \;\; \mathbb{M} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}$$

and :

$$K(p,x) = \begin{pmatrix} p\varepsilon(p,x) & 0 \times \mathbb{I}_3 \\ 0 \times \mathbb{I}_3 & p\mu(p,x) \end{pmatrix} \in Hom(\mathbb{C}^6),$$

$$\forall u \in \mathbb{C}^3, Re \left\langle (\Lambda + \Lambda^*)u, \overline{u} \right\rangle \geq \alpha |u|^2.$$

Main problem because of metamaterial

Main problem

Physical parameters of metamaterials depends on the frequency and could become negative definite on the behavior of some p.
$\Longrightarrow$ The multiplicative operator $K(p)$ is no longer coercive for some p.

Goal

a) Give conditions on the homogenized parameters of the metamaterials leading to the well-posedness of the Maxwell's system.
b) Look for numerical scheme suitable for metamaterials.

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An example : periodic array of SRR

B. Kanté, SN Burokur, F. Gadot, and A. de Lustrac. Métamatériau à indice de réfraction négatif en infrarouge.

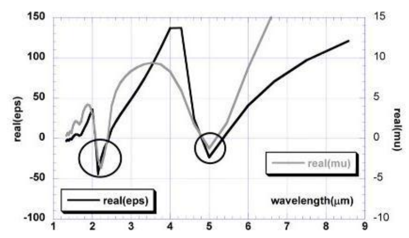

Fig.: Periodic array of S.R.R

Homogenized parameters

$$[\varepsilon(p,x)] = \left(1 + \frac{\omega_p^2}{p^2}\right)\mathbb{I}_3, \quad [\mu(p,x)] = \left(1 + \frac{\delta p^2}{-p^2 - \omega_0^2 + p\Gamma}\right)\mathbb{I}_3.$$

Main result

Main result

Assumptions on the metamaterial :

- (p-Regularity) $p \in D_0 \longrightarrow K(p,x)$ is holomorphic for almost all $x \in \Omega$.
- (x-regularity) $x \in \overline{\Omega} \longrightarrow K(p,x)$ belong to $L^\infty(\overline{\Omega})$ for all $p \in D_0$.
- (Classical) There exist $p_0 \in D_0$ such that $Re\left(K(p_0) + K(p_0)^*\right) \succeq \alpha$.
- (technical...) There exist $a \in Lip(\overline{\Omega}, \mathcal{O}(D_0))$ such that $Re\left(a(p)K(p) + a(p)^* K(p)^*\right) \succeq \alpha$.

Theorem

Assume that (p-Regularity)-(x-Regularity)-(Classical)-(technical...) are satisfied. Then the Maxwell's system in presence of metamaterial is well-posed for all $p \in D_0 \backslash S$ where S is a discrete set of D_0. Moreover, one has the estimate :

$$\left\{\int_\Omega |E(p,x)|^2 + |H(p,x)|^2 dx\right\}^{\frac{1}{2}} \lesssim \left\{\int_\Omega \left|(K(p,x) + \mathbb{M})\begin{pmatrix} E(p,x) \\ H(p,x)\end{pmatrix}\right|^2 dx\right\}^{\frac{1}{2}}$$

Numerical simulation

Idea : From [A.K. Aziz, S. Leventhal, Finite element approximation for first order systems, SIAM J Num An, 1978.] $\Rightarrow$ Limited to coercive operators.

Variational formulation :

$$\textit{Find } u_h = (E_h, H_h) \in \mathcal{V}_h \subset \mathcal{D}(\mathbb{M}) \textit{ such that } \forall \psi_h \in (K(p,x) + \mathbb{M})\, \mathcal{V}_h :$$

$$\int_\Omega \langle (K(p,x) + \mathbb{M})\, u_h, \overline{\psi_h} \rangle \, dx = \int_\Omega \langle f, \overline{\psi_h} \rangle \, dx$$

Theorem

Assume that $\forall u \in H^s(\Omega) \cap \mathcal{D}(\mathbb{M})$ *there exists* $\widetilde{u}_h \in \mathcal{V}_h$ *such that*

$$\| u - \widetilde{u}_h \|_{\mathcal{D}(\mathbb{M})} \leq C h^s \| u \|_{H^s(\Omega) \cap \mathcal{D}(\mathbb{M})} .$$

Then if the solution $u \in H^s(\Omega) \cap \mathcal{D}(\mathbb{M})$ *one has the error inequality :*

$$\| u - u_h \|_{L^2(\Omega)^6} \leq C h^s \| u \|_{H^s(\Omega) \cap \mathcal{D}(\mathbb{M})} .$$

Proof part (1)

- Let $\mathcal{A}(u,v) = \int_{\Omega} \left\langle (K(p,x)+\mathbb{M})\, u, \overline{(K(p,x)+\mathbb{M})\, v} \right\rangle dx$. From the theorem, one gets :

$$\forall u \in \mathcal{D}(\mathbb{M}),\ \mathcal{A}(u,u) \geq C \parallel u \parallel^2_{L^2(\Omega)} .$$

- Let u_h satisfying $\mathcal{A}(u_h, \phi_h) = \int_{\Omega} \left\langle f, \overline{\phi_h} \right\rangle dx$. Since $dim(\mathcal{V}_h) < \infty$ and $\mathcal{V}_h \subset \mathcal{D}(\mathbb{M})$, u_h is unique.
- Let $\widetilde{u}_h \in \mathcal{V}_h$ such that

$$\parallel u - \widetilde{u}_h \parallel_{\mathcal{D}(\mathbb{M})} \leq C h^s \parallel u \parallel_{H^s(\Omega)\cap\mathcal{D}(\mathbb{M})} .$$

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Proof part (2)

- For all $v_h \in \mathcal{V}_h$ we have :

$$\mathcal{A}(u - \widetilde{u}_h, v_h) = \mathcal{A}(u_h - \widetilde{u}_h, v_h).$$

- Taking $v_h = u_h - \widetilde{u}_h$ and using the Schwartz inequality, one gets :

$$\begin{aligned} \| \, (K(p,x) + \mathbb{M}) \, (u_h - \widetilde{u}_h) \, \|_{L^2(\Omega)} \quad & \leq C \, \| \, (K(p,x) + \mathbb{M}) \, (u - \widetilde{u}_h) \, \|_{L^2(\Omega)} \\ & \leq C \, \| \, u - \widetilde{u}_h \, \|_{\mathcal{D}(\mathbb{M})} \\ & \leq Ch^s \, \| \, u \, \|_{H^s(\Omega) \cap \mathcal{D}(\mathbb{M})} \end{aligned}$$

- Then, from the coercivity inequality, it follows :

$$\| \, u - u_h \, \|_{L^2(\Omega)} \leq Ch^s \, \| \, u \, \|_{H^s(\Omega) \cap \mathcal{D}(\mathbb{M})}$$

- $\Omega =]a, b[\subset \mathbb{R}$.

$$\left\{ \begin{array}{l} \textit{Find } (u, v) \in H_0^1(\Omega) \times H^1(\Omega) \textit{ such that :} \\ \qquad p\varepsilon(p, x)u - \partial_x v = f_1, \\ \qquad p\mu(p, x)v - \partial_x u = f_2 \end{array} \right. \tag{1}$$

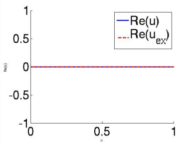
Remark :

Problem well-posed if (x-p-regularity) and ("Classical") hold.

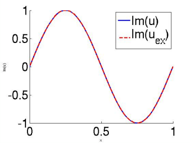
An exact solution :

- $\Omega =]0, 1[$, $f_1(x) = ip\varepsilon \sin(2\pi x) + \sin(x)$, $f_2(x) = p\mu \cos(x) - 2i\pi \cos(2\pi x)$.
- $u(x) = i \sin(2\pi x)$, $v(x) = \cos(x)$.
- $p = iw$, $w = 15$.

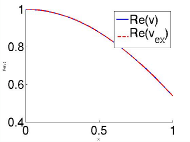
Numerical experiments (1) : the vaccum



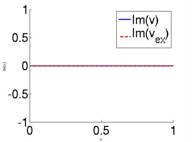
(a) Re(u)



(b) Im(u)

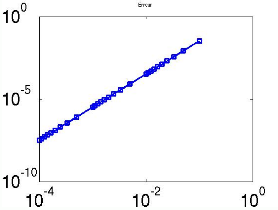


(c) Re(v)

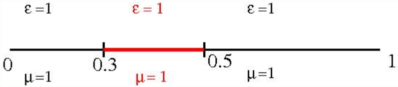


(d) Im(v)

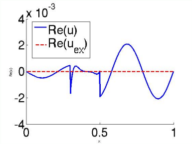
Fig.: The vacuum



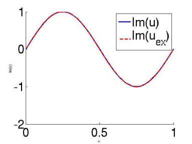
(a) Error, $\| u - u_h \|_{L^2}$



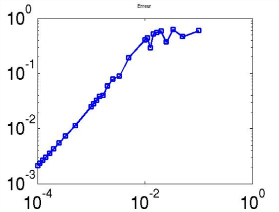
(b) Geometrie



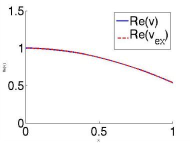
(c) Re(u)



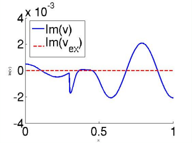
(d) Im(u)



(a) Error, $\| u - u_h \|_{L^2}$

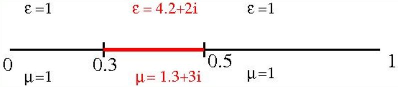


(e) Re(v)



(f) Im(v)

Fig.: A dielectric media



(b) Geometrie

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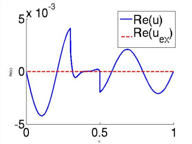
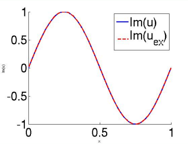
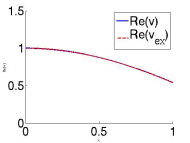
Numerical experiments (3) : A metamaterial



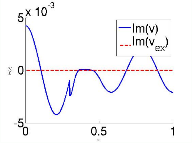
(c) Re(u)



(d) Im(u)

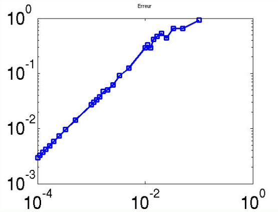


(e) Re(v)



(f) Im(v)

Fig.: A metamaterial



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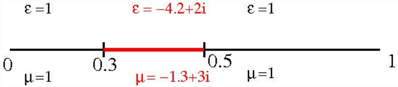

(b) Geometrie

Conclusions and prospects

Conclusion :

- smooth metamaterials $\Longrightarrow$ well-posed problems.
- Numerical approximation $\Longrightarrow$ OK with some kind of finite element method.

Prospects :

- Study of the well-posedness for L^∞ metamaterials (transmission problems???).
- Convergence of Discontinuous Galerkin methods.
- Numerical test in 2D.
- ....

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Thank you for you attention.