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ON THE EXISTENCE AND UNIQUENESS OF A SOLUTION FOR SOME FREQUENCY-DEPENDENT PARTIAL DIFFERENTIAL EQUATIONS COMING FROM THE MODELING OF METAMATERIALS*

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Abstract. Several systems coming from the theory of linear wave propagation are investigated, on a bounded domain, in the presence of frequency-dependent materials like metamaterials. For each system we show generic well-posedness results under assumptions that are relevant for some models in the literature. This means existence and uniqueness of a solution for all frequencies except for a discrete locally finite and possibly empty set of frequencies. Finally, some examples of materials are studied, like a periodic array of split-ring resonators (SRR), a chiral metamaterial based on the Ω-particle resonator model, a bi-anisotropic metamaterial made from SRR, absorbing boundary conditions of perfectly matched layers type for the acoustics waves, an example of acoustic metamaterial having negative bulk modulus and an elastic metamaterial.

Key words. metamaterials, Maxwell's equations, wave equation, linear elasticity

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1. Introduction.

In 1968, Veselago [34] theoretically investigated the effects of electromagnetic and acoustic phenomena in materials having simultaneously negative values for the permittivity ε and the permeability μ. As they reverse the Poynting vector, they were called "left-handed materials" (LHM). Veselago also noticed that they have exotic properties like a reversed Doppler effect or a reversed Vavilov–Cerenkov radiation effect. The keen interest in LHM was initiated by Pendry in 2000 [29, 30]. He managed to build LHM using a periodic array of split-ring resonators (SRR). Thus these exotic structures were named "metamaterials." There are various planned applications of metamaterials such as the perfect lens [29], sound focusing [16], cloaking effect using the concept of transformation optics [5, 14], or controlling light using photonic crystals [28].

However, as metamaterials cannot be found in nature as negative indexes materials, they are usually seen as the result of a frequency-dependent homogenization (see, for example, [11, 17, 32, 33]). The homogenized parameters which may depend on the pulsation w become negative definite for some w. However, the usual framework to prove the well-posedness of the systems modeling electromagnetics, acoustics, or elastodynamics wave propagation in materials assumes the parameters to be positive definite. The question we thus want to ask is, What happen for the existence and uniqueness of solutions to the equations mentioned above in presence of metamaterials?

Another way to treat these problems should be to consider minimization variational principles like those studied in [25] for the acoustics, elastodynamics, and electromagnetism in lossy inhomogeneous bodies at fixed frequency. These principles rely on the introduction of a saddle point minimization problem in order to study, for instance, Maxwell's equation. However, this paper does not give existence and

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uniqueness of solution to Maxwell's equation. Moreover, to be applied, a special decomposition of the physical parameters have to be achieve in order to get some positive definite tensors that we were not able to exhibit in the case of negative index materials. An additional way to deal with the well-posedness of partial differential equations in the presence of metamaterials is to use some fixed-point methods like those introduced in [7], wherein Maxwell's equation in the presence of some chiral media are solved. However, the development of this method has only been done in case of variable but positive parameters and thus does not extend to the case of metamaterials. Consequently, we are not able to use such methods to study existence and uniqueness of a solution for systems of partial differential equation with sign-changing coefficients.

On the subject of the mathematical study of problems involving metamaterials, there exists, to the best of the authors' knowledge, only a few works. For scalar equations in presence of media having sign-shifting coefficients, some studies were done in [2, 3]. To be more precise, these papers study a second order scalar problem with parameters that does not depend on the frequency for the transmission problem between a "classical" material and a "negative" one. Then, using an integral equation technique on the interface between the two media, they manage to prove that the considered problem can be solved with the Fredholm alternative and they give a new way to approximate the solution of these kinds of problems [1]. In [3], the authors use the so-called T-coercivity (or inf-sup) approach to show that if the interface between the "classical" material and the metamaterial is smooth, then the problem relies once again on the Fredholm alternative. They can even build in some particular geometric setting the operator T leading to the existence and uniqueness of a solution. For works dealing with second order vector equations, we can cite [12], where Maxwell's equation in the presence of bi-anisotropic material and metamaterials is studied. They give conditions on the materials that allow the use of the Lax–Milgram theorem and prove the convergence of the classical finite element method within this framework.

However, from a remark of Veselago [34], the parameters of a metamaterial have to depend on the pulsation w and could become negative definite only for some w in a bounded set. The motivation of a frequency dependence for the homogenized metamaterials comes also from the ability to describe a material coming from a well-posed system on a time domain from an energy point of view. Furthermore, one can notice (see, for instance, [21, 32, 35]) that the homogenized coefficients of such materials usually depend on the frequency like rational fractions. As a result they can be considered as a holomorphic function of $p = iw + \delta$ on some connected open set of $\mathbb{C}$. Actually, it is worth noting that these observations have been confirmed by experiments performed on several metamaterials obtained from a homogenization process [17, 21, 32, 33, 10, 36, 23]. Note that the previously mentioned well-posedness results are established for metamaterials that do not depend on the frequency. Nevertheless, these results can be used to study frequency-dependent metamaterials having specific physical parameters for $w = w_0$. However, the main tool to obtain well-posedness of frequency-dependent partial differential equations is the Fredholm analytical theory (see [19, p. 371]), which discards an unknown set of frequencies which is not a priori empty and hard to determine (see, for instance, [22]). Thus we cannot state a priori the existence and uniqueness of solution for frequency-dependent metamaterials at w_0, whatever the geometry is, since it could happen that w_0 has been discarded. Consequently we cannot use the previously mentioned results to straightforwardly study the examples mentioned above.

In this paper, we introduce a frequency dependence of the physical parameters invoked in first order symmetric systems describing wave propagation through electromagnetic, acoustic, or linear elastic metamaterials. More precisely, a global rather than local approach in the frequency domain is considered and our goal is to provide well-posedness for the frequency-dependent cases mentioned. Unfortunately, we will not be able to prove such results for all frequencies. Instead we formulate generic well-posedness that has existence and uniqueness except for a discrete and locally finite set of frequencies. The key tool for the proof of the presented results is the Fredholm theory, which removes some frequencies depending on what the geometry and the boundary conditions are. Consequently, our results are not global well-posedness. Nevertheless, we are going to give here sufficient and general conditions on the materials leading to discreteness and local finiteness for the singular frequencies to be dropped out.

An outline of this article is as follows. First, a mathematical framework to study Maxwell's equation, the acoustic wave system, and linear elasticity is introduced to show the general underlying difficulties due to the presence of metamaterials (section 2). We then introduce the Maxwell equations we are going to study and give two instances of generic well-posedness for these equations that are not equivalent (section 3). The first one (section 3.1) can be used to study the electromagnetic effect in scalar chiral or bi-anisotropic metamaterials. The second one (section 3.2) can handle Maxwell's equation in the presence of anisotropic metamaterials. Following the same sketch, we derive generic well-posedness results respectively for acoustics (section 4) and elastic metamaterials (section 5). Finally (section 6), in order to show the interest of these results, we will apply them to study some physical examples of metamaterials.

2. General mathematical framework. We present here in a very general setting the difficulties arising from the presence of metamaterials and the main tools used in the proof of our results.

Let Ω be a simply connected and bounded open set of $\mathbb{R}^N$ with $\mathcal{C}^1$ boundary. The exterior unit normal vector is denoted by $\nu = (\nu_1, \nu_2, \nu_3)$. Let $\mathbb{S} = \sum_{j=1}^{N} \mathcal{S}_j \partial_j$ be a first order differential operator where $\mathcal{S}_j \in \mathrm{Hom}(\mathbb{C}^k)$ are linear applications from $\mathbb{C}^k$ to $\mathbb{C}^k$ satisfying $\mathcal{S}_j^* = \mathcal{S}_j$. We now consider the following system:

$$\text{(2.1)}\qquad \begin{array}{l} \textit{Find } u \in \mathcal{H}_{\mathbb{S}} \textit{ such that} \\ \left\{ \begin{array}{l} K(p,x)u + \mathbb{S}u(p,x) = f(x), \; x \in \Omega, \\ u(p,x) \in \ker N(x), \; x \in \partial\Omega, \end{array} \right. \end{array}$$

where $p = iw + \delta$ is the Laplace variable, $N(x)$ is a smooth varying linear application, f is a source term, $K(p,x) \in \mathrm{Hom}(\mathbb{C}^k)$, $u(x) \in \mathbb{C}^k$, is an unknown physical quantity (electric or magnetic fields, for instance), and $\mathcal{H}_{\mathbb{S}} = \left\{ u \in L^2(\Omega)^k \mid \mathbb{S}u \in L^2(\Omega)^k \right\}$. We note $\mathcal{D}(\mathbb{S})$ for the domain of the unbounded operator $\mathbb{S}$:

$$\mathcal{D}(\mathbb{S}) = \{ u \in \mathcal{H}_{\mathbb{S}} \mid u(x) \in \ker N(x) \text{ for } x \in \partial\Omega \} .$$

The boundary conditions are assumed to verify the conditions given in [31] to ensure maximal dissipativity of the unbounded operator $(\mathbb{S}, \mathcal{D}(\mathbb{S}))$. As a consequence, when the multiplicative operator $K(p, .)$ is (uniformly in the second variable) bounded and coercive, (2.1) has a unique solution continuously depending on the source term.

However, when metamaterials are strictly embedded into Ω, the application $K(p,x)$ can no longer be coercive (nor have any specific sign as x travels through Ω) for some p. Hence we cannot deduce existence and uniqueness of a solution for (2.1) from the previous argument. This paper is thus devoted to proving well-posedness of systems modeling wave propagation through metamaterials in electromagnetism or optics

(Theorem 3.2), acoustics (Theorem 4.1), and linear elasticity (Theorem 5.1) when the tensor $K(p,x)$ verifies the following hypotheses.

Assumption 1 (general assumptions).

(H1) The application $p \in D_0 \longmapsto K(p,x)$ is holomorphic for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(H2) The application $x \in \overline{\Omega} \longmapsto K(p,x)$ belongs to $L^\infty(\overline{\Omega})$ (or is replaced by Lipschitz continuous, which will be later specified) for all $p \in D_0$ and $K(p,x)^{-1}$ exists for almost all $x \in \overline{\Omega}$.

(H3) There exists p_0 in D_0 such that $K(p_0,.)$ is coercive.

The above assumptions are the basics of our framework for studying system (2.1) with metamaterials. (H1) describes the frequency dependence of the physical parameters of the metamaterials. Note that for parameters admitting poles in p, we can restrict D_0 to a subset of the complementary of the union of neighborhoods of these poles. (H2) is nothing but a physical assumption saying that the homogenized metamaterials have smoothly varying (in $x \in \Omega$) physical parameters. (H3) says that the material behaves like a "physical" one (that is, when the parameters are positive definite) for a given $p = p_0$. Furthermore, (H1) and (H3) are motivated by the remark of Veselago [34] mentioned in the introduction and some examples extracted from the literature [17, 32, 10]. In particular, these hypotheses yield the well-posedness of (2.1) for p_0; see [31].

Let us present now the sketch of the proof of the results presented here. The basic idea behind the method used to bypass the difficulty relying on the "negativity" of $K(p,.)$ is to find a way to extend the existence and uniqueness of solution to (2.1) from p_0 to other p lying in the holomorphy domain of $K(p,.)$. The main tool to do so can be found in the Fredholm's analytical theory [19, p. 371].

First, for the systems we work with, there exists a first order differential operator $\mathbb{Q}$ such that $\mathbb{Q}\mathbb{S} = 0$ in the sense of distribution and $\ker(\mathbb{S}) = \overline{\mathrm{Im}(\mathbb{Q}^*)}$, where $\ker(\mathbb{S}) = \{u \in L^2(\Omega)^k \mid \mathbb{S}u = 0,\ \mathcal{D}'(\Omega)\}$. Moreover, they are shown to be subjected to some coercive inequality [24] of the form

$$\|u\|_{H^1(\Omega)} \le C\left\{\|u\|_{L^2(\Omega)} + \|\mathbb{S}u\|_{L^2(\Omega)} + \|\mathbb{Q}u\|_{L^2(\Omega)}\right\}, \tag{2.2}$$

where $u \in \mathcal{D}(\mathbb{S}) \cap \{u \in L^2(\Omega)^k \mid \mathbb{Q}u \in L^2(\Omega)\}$. Then, to obtain some compactness for the resolvent of $(\mathbb{S}, \mathcal{D}(\mathbb{S}))$ one has to control $\mathbb{Q}u$ for $u \in \mathcal{D}(\mathbb{S})$. One way to proceed is to use an orthogonal splitting of $L^2(\Omega)^k$ given by a suitable Hodge decomposition [9, 4]:

$$L^2(\Omega)^k = \mathcal{R} \oplus \mathcal{R}^\perp,$$

where $\mathcal{R}$ is a linear subspace of $L^2(\Omega)^k$ such that $\ker(\mathbb{S}) \subset \mathcal{R}$. This implies that $\ker(\mathbb{S})^\perp = \ker(\mathbb{Q}) \supset \mathcal{R}^\perp$. With such a decomposition at hand, one can project (2.1) to get the following equivalent system:

$$\begin{array}{l} \textit{Find } u = P_{\mathcal{R}}u + P_{\mathcal{R}^\perp}u \in \mathcal{D}(\mathbb{S}) \textit{ such that} \\ \left\{ \begin{array}{l} P_{\mathcal{R}^\perp}K(p,.)(P_{\mathcal{R}}u + P_{\mathcal{R}^\perp}u) + P_{\mathcal{R}^\perp}\mathbb{S}P_{\mathcal{R}^\perp}u = P_{\mathcal{R}^\perp}f, \text{ on } \Omega, \\ P_{\mathcal{R}}K(p,.)(P_{\mathcal{R}}u + P_{\mathcal{R}^\perp}u) = P_{\mathcal{R}}f, \text{ on } \Omega, \end{array} \right. \end{array}$$

where $P_{\mathcal{R}} : L^2(\Omega)^k \longrightarrow \mathcal{R}$ and $P_{\mathcal{R}^\perp} : L^2(\Omega)^k \longrightarrow \mathcal{R}^\perp$ are the orthogonal projections associated to the Hodge decomposition. Note then that one needs to invert the family of operators $P_{\mathcal{R}^\perp}K(p,.)P_{\mathcal{R}^\perp} + P_{\mathcal{R}^\perp}\mathbb{M}P_{\mathcal{R}^\perp}$ on $\mathcal{R}^\perp \cap \mathcal{D}(\mathbb{S})$. As this family is holomorphic (thanks to (H1)) and compact resolvent (thanks to inequality (2.2)), this inversion can be done with the help of Fredholm analytic theory. This yields to existence and uniqueness of $P_{\mathcal{R}^\perp}u(p,.)$ for all $p \in D_0 \backslash S$ for S a set of exceptional values. However, what is difficult is to get $P_{\mathcal{R}}u$ in this very general setting. Indeed, one has to invert the operator $P_{\mathcal{R}}K(p,.)P_{\mathcal{R}}$ on $\mathcal{R}$ for all $p \in D_0$, even when $K(p,.)$ is not coercive. To do this, one must have more information on the space $\mathcal{R}$, implying that we must specify the physics we work with. We thus prove Theorems 3.2, 4.1, and 5.1 but under restrictive phenomenological assumptions (scalar valued physical parameters, for instance). In some others cases we can prove the invertibilty of $P_{\mathcal{R}}K(p,.)P_{\mathcal{R}}$ (leading to Theorems 3.6, 4.5, and 5.2) by adding one hypothesis of the following form.

Assumption 2 (general additional assumptions for nonscalar parameters).

(H4) There exists $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, such that $K(p,x)a(p,x)$ is coercive for all $p \in D_0$ and almost all $x \in \Omega$.

Remark 2.1. Assumption (H4) is added when tensorial and not scalar parameters are considered because of change of the spectral properties of the operator involved. For instance, Maxwell's equations in chiral media may admit an essential spectrum close to zero on the positive imaginary axis. This does not occur in isotropic media [22]. Finally note that assumption (H4) is satisfied for scalar Lipchitz continuous parameters satisfying (H1)–(H2). Indeed, in that case, (H4) is satisfied with $a(p,x) = K(p,x)^{-1}$.

3. Generic well-posedness results for electromagnetism and optics.

In this section, we study electromagnetic phenomena in the presence of metamaterials with three different models. First we look at the usual time-harmonic Maxwell's equations in Laplace transform:

$$
\text{(3.1)} \qquad \begin{array}{l} \textit{Find } (e,h) \in H(\mathrm{curl},\Omega)^2 \textit{ such that} \\ \left\{ \begin{array}{l} p\varepsilon(p,x)e(p,x) - \nabla \times h(p,x) = -j(x) \text{ on } \Omega, \\ p\mu(p,x)h(p,x) + \nabla \times e(p,x) = -m(x) \text{ on } \Omega, \\ \nu(x) \times (e(p,x) + \Lambda(x)(\nu(x) \times h(p,x))) = 0, \ x \in \partial\Omega, \end{array} \right. \end{array}
$$

where $H(\mathrm{curl},\Omega) = \left\{\Psi \in L^2(\Omega)^3 \mid \nabla \times \Psi \in L^2(\Omega)^3\right\}$, e is the electric field, h is the magnetic field, ε stands for the permittivity, μ stands for the permeability, and j and m are respectively the electric and magnetic current densities. At last $\Lambda : \partial\Omega \longrightarrow \mathrm{Hom}(\mathbb{C}^3)$ describes impedance boundary condition and is assumed to be Lipschitz continuous and coercive:

$$
\forall \ z \in \mathbb{C}^3, \mathcal{R}e \left\langle (\Lambda + \Lambda^*) \, z, \overline{z} \right\rangle \geq 0,
$$

noting $\langle X, Y \rangle = \sum_{j=1}^N X_j Y_j$ the standard scalar product of vector $X, Y \in \mathbb{C}^N$ with associated norm $|X| = \sqrt{\langle X, X \rangle}$. Note that the tangential traces on $\partial\Omega$ of $(e,h) \in H(\mathrm{curl},\Omega)^2$ have to be understood in the sense that $\left(\nu \times e_{|\partial\Omega}, \nu \times h_{|\partial\Omega}\right) \in H^{-\frac{1}{2}}(\partial\Omega)^2$.

A second model we are interested in describes the wave propagation of time-harmonic electromagnetic waves through bi-anisotropic media [21]:

$$
\text{(3.2)} \qquad \begin{array}{l} \textit{Find } (e,h) \in H(\mathrm{curl},\Omega)^2 \textit{ such that} \\ \left\{ \begin{array}{l} p\varepsilon(p,x)e(p,x) + p\xi(p,x)h(p,x) - \nabla \times h(p,x) = -j(x) \text{ on } \Omega, \\ p\mu(p,x)h(p,x) + p\zeta(p,x)e(p,x) + \nabla \times e(p,x) = -m(x) \text{ on } \Omega, \\ \nu(x) \times (e(p,x) + \Lambda(x)(\nu(x) \times h(p,x))) = 0, \ x \in \partial\Omega, \end{array} \right. \end{array}
$$

where ξ and ζ are called the coupling constants.

Finally, in the special case of chiral media appearing in optics for crystals, one can replace Maxwell's equations by the Drude–Born–Fedorov system [17]:

$$\text{(3.3)}\qquad \begin{array}{l} \textit{Find } (e,h) \in H(\mathrm{curl},\Omega)^2 \textit{ such that} \\ \left\{ \begin{array}{l} p\varepsilon(p,x)e(p,x) + p\beta(p,x)\varepsilon(p,x)\nabla\times e(p,x) \\ \qquad\qquad - \nabla\times h(p,x) = -j(x),\ x\in\Omega, \\ p\mu(p,x)h(p,x) + p\beta(p,x)\mu(p,x)\nabla\times h(p,x) \\ \qquad\qquad + \nabla\times e(p,x) = -m(x),\ x\in\Omega, \\ \nu(x)\times(e(p,x) + \Lambda(x)(\nu(x)\times h(p,x))) = 0,\ x\in\partial\Omega, \end{array} \right. \end{array}$$

where β is the chirality of the material embedded into Ω.

Remark 3.1. Maxwell's equations (3.1) are a special case of (3.2) and (3.3) that can be obtained respectively by taking $\beta = 0$ or $\xi = \zeta = 0$.

Moreover, (3.2) and (3.3) are equivalent up to some algebraic computations. Hence, we will give two sets of assumptions on the media embedded into Ω according to whether it is scalar bi-anisotropic (Assumption 3) or chiral (Assumption 4), but only one theorem will be formulated (Theorem 3.2). To show that (3.3) is similar to (3.2) (the reciprocal can be done in the same way), first rewrite (3.3) as follows:

$$\left\{ \begin{array}{l} \nabla\times h = p\varepsilon e + p\beta\varepsilon\nabla\times e + j, \\ \nabla\times e = -m - p\mu h - p\beta\mu\nabla\times h, \end{array} \right.$$

where the dependence of the physical parameters in (p,x) has been dropped to lighten the overall expressions. Then, reporting the first equation above into the second one of (3.3) and vise versa, one gets

$$\left\{ \begin{array}{l} p\varepsilon e + p\beta\varepsilon\left(-m - p\mu h - p\beta\mu\nabla\times h\right) - \nabla\times h = -j, \\ p\mu h + p\beta\mu\left(p\varepsilon e + p\beta\varepsilon\nabla\times e + j\right) + \nabla\times e = -m. \end{array} \right.$$

Finally, gathering the previous computations, one has the following (3.2) system:

$$\left\{ \begin{array}{l} \left(\mathbb{I}_3 + p^2\beta^2\varepsilon\mu\right)^{-1}\left(p\varepsilon e - p^2\beta\varepsilon\mu h\right) - \nabla\times h = -\widetilde{j}, \\ \left(\mathbb{I}_3 + p^2\beta^2\mu\varepsilon\right)^{-1}\left(p\mu h + p^2\beta\mu\varepsilon e\right) + \nabla\times e = -\widetilde{m}. \end{array} \right.$$

We are going to give two generic well-posedness results (that is, existence and uniqueness of a solution except for some p) for electromagnetism. The first generic well-posedness result allows us to study either chiral (3.3) or bi-anisotropic (3.2) materials having scalar physical parameters satisfying assumptions similar to (H1), (H2), (H3). The second result, for 3×3 tensorial coefficients, requires additionally a variant of assumption (H4).

3.1. Generic well-posedness for scalar chiral or bi-anisotropic materials. We focus here on solving (3.3) or (3.2) in the presence of materials characterized by scalar parameters, and we introduce the following assumptions.

Assumption 3 (for scalar bi-anisotropic materials).

(B1) The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$, and $\zeta(p,x)$ are holomorphic in $p\in D_0$ for almost all $x\in\Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(B2) The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$, and $\zeta(p,x)$ are Lipschitz continuous in $x\in\overline{\Omega}$ for all $p\in D_0$. Moreover, $\varepsilon(p,x)\mu(p,x) - \xi(p,x)\zeta(p,x) \neq 0$ for almost all $x\in\overline{\Omega}$ and for any $p\in D_0$.

(B3) There exists p_0 in D_0 and $\alpha > 0$ such that the following inequality holds:

$$\mathcal{R}e\left(\langle p_0\varepsilon(p_0,x)X,\overline{X}\rangle + \langle p_0\mu(p_0,x)Y,\overline{Y}\rangle + \langle p_0\xi(p_0,x)Y,\overline{X}\rangle\right) + \mathcal{R}e\left(\langle p_0\zeta(p_0,x)X,\overline{Y}\rangle\right) \geq \alpha(|X|^2+|Y|^2)$$

for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

Assumption 4 (for scalar chiral media).

(C1) The applications $\varepsilon(p,x), \mu(p,x)$, and $\beta(p,x)$ are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(C2) The applications $\varepsilon(p,x), \mu(p,x)$, and $\beta(p,x)$ are Lipschitz continuous in $x \in \overline{\Omega}$ for all $p \in D_0$. Moreover, $p^2\varepsilon(p,x)\mu(p,x)M(p,x) \neq 0$ with $M(p,x) = (1 + p^2\beta(p,x)^2\varepsilon(p,x)\mu(p,x))^{-1}$ for almost all $x \in \overline{\Omega}$.

(C3) There exist p_0 in D_0 and $\alpha > 0$ such that for almost all $x \in \Omega$ and $X, Y \in \mathbb{C}^3$,

$$\begin{aligned}&\mathcal{R}e\left\{\langle p_0\varepsilon(p_0,x)M(p_0,x)X,\overline{X}\rangle + \langle p_0\mu(p_0,x)M(p_0,x)Y,\overline{Y}\rangle\right\}\\ &+ \mathcal{R}e\left\{\langle p_0^2\beta(p_0,x)\varepsilon(p_0,x)\mu(p_0,x)M(p_0,x)Y,\overline{X}\rangle\right\}\\ &- \mathcal{R}e\left\{\langle p_0^2\beta(p_0,x)\varepsilon(p_0,x)\mu(p_0,x)M(p_0,x)X,\overline{Y}\rangle\right\} \geq \alpha(|X|^2+|Y|^2),\end{aligned}$$

Finally, let us note $H(\text{div},\Omega) = \left\{e \in L^2(\Omega)^3 \mid \text{div}e \in L^2(\Omega)\right\}$. Then, the first main result of this paper is the following theorem.

THEOREM 3.2. *Suppose that $(j,m) \in (H(\text{div},\Omega))^2$ and that Assumption* 3 *(respectively* 4*) is satisfied. Then Maxwell's equations* (3.2) *(respectively,* (3.3)*) are well-posed for all $p \in D_0\backslash S$, where S is a discrete, locally finite, and possibly empty set of D_0. Moreover the solution $(e(p,.),h(p,.))$ verifies*

$$\|(e(p,.),h(p,.))\|_{L^2(\Omega)^6} \leq C(p)\left\{\left\|\begin{pmatrix} j \\ m\end{pmatrix}\right\|_{L^2(\Omega)^6} + \left\|\begin{pmatrix}\text{div}(j)\\ \text{div}(m)\end{pmatrix}\right\|_{L^2(\Omega)^2}\right\}$$

with $C(p)$ a constant depending only on p and Ω. Moreover, the application $p \in D_0\backslash S \longmapsto (e(p,.),h(p,.)) \in L^2(\Omega)^6$ is holomorphic.

Proof. The proof follows the sketch presented in section 2. Hence we start by formulating Maxwell's equations as

$$\text{(3.4)}\qquad \begin{aligned}&\textit{Find } u = (e,h)^T \in \mathcal{H}_{\mathbb{M}} \textit{ such that}\\ &\begin{cases} K(p,x)u(p,x) + \mathbb{M}u(p,x) = f(x), \; x \in \Omega,\\ \nu(x) \times (e(p,x) + \Lambda(x)(\nu(x) \times h(p,x))) = 0, \; x \in \partial\Omega,\end{cases}\end{aligned}$$

where $f = (-j,-m) \in H(\text{div},\Omega)^2$ and $\mathbb{M}$ is the unbounded operator

$$\mathbb{M} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0\end{pmatrix}$$

with domain $\mathcal{D}(\mathbb{M})$ defined by

$$\mathcal{D}(\mathbb{M}) = \left\{(e,h)^T \in (H(\text{curl},\Omega))^2 \mid \nu(x) \times \left(e_{|\partial\Omega} + \Lambda(x)(\nu(x) \times h_{|\partial\Omega})\right) = 0, \; x \in \partial\Omega\right\}.$$

Note that $(\mathbb{M},\mathcal{D}(\mathbb{M}))$ is maximal dissipative. Finally, $K(p,x) \in \text{Hom}(\mathbb{C}^6)$ is given by

$$\text{(3.5)}\qquad K(p,x) = \begin{pmatrix} K_{11}(p,x)\mathbb{I}_3 & K_{12}(p,x)\mathbb{I}_3 \\ K_{21}(p,x)\mathbb{I}_3 & K_{22}(p,x)\mathbb{I}_3\end{pmatrix},$$

where $\mathbb{I}_N$ denotes the identity operator of $\mathbb{C}^N$, and where

- for bi-anisotropic media (3.2), $K_{11}(p,x) = p\varepsilon(p,x)$, $K_{12}(p,x) = p\xi(p,x)$, $K_{21}(p,x) = p\zeta(p,x)$, $K_{22}(p,x) = p\mu(p,x)$;
- for chiral media (3.3), $K_{11}(p,x) = p\varepsilon(p,x)M(p,x)$; $K_{21}(p,x) = -K_{12}(p,x)$, $K_{12}(p,x) = p^2\beta(p,x)\varepsilon(p,x)\mu(p,x)M(p,x)$, $K_{22}(p,x) = p\mu(p,x)M(p,x)$, where $M(p,x)$ is defined in (C3).

Note $\mathbb{Q} = \left(\begin{smallmatrix}\mathrm{div} & 0\\ 0 & \mathrm{div}\end{smallmatrix}\right)$. Then, using that $\mathbb{QM} = 0$ and that $K(p,.)$ satisfies (B2) allows us to take the divergence of (3.4) to obtain

$$\mathbb{Q}f = \mathbb{Q}K(p,x)u + \mathbb{QM}u \quad =: Z(p,x)u + \widetilde{K}(p,x)\mathbb{Q}u.$$

In this last identity, $\widetilde{K}(p,x) \in \mathrm{Hom}(\mathbb{C}^2)$ and $Z(p,x) \in \mathrm{Hom}(\mathbb{C}^6, \mathbb{C}^2)$ are given by

$$\widetilde{K}(p,x) = \begin{pmatrix} K_{11}(p,x) & K_{12}(p,x) \\ K_{21}(p,x) & K_{22}(p,x) \end{pmatrix},$$
$$Z(p)u = Z(p)\begin{pmatrix} e \\ h \end{pmatrix} = \begin{pmatrix} \langle \nabla K_{11}(p,x), e\rangle + \langle \nabla K_{12}(p,x), h\rangle \\ \langle \nabla K_{21}(p,x), e\rangle + \langle \nabla K_{22}(p,x), h\rangle \end{pmatrix}.$$

An elliptization of Maxwell's equations is then performed by writing a relaxed version of system (3.4) by the introduction of two new unknown functions φ and ψ—identically zero in (3.4)—as follows [22]:

$$\textit{Find } (e,h,\varphi,\psi) \in \mathcal{D}(\mathbb{M}) \cap (H(\mathrm{div},\Omega))^2 \times H^1_0(\Omega)^2 \textit{ such that}$$
$$\begin{cases} (K(p,x) + \mathbb{M})\begin{pmatrix} e \\ h \end{pmatrix} + \begin{pmatrix} \nabla\varphi \\ \nabla\psi \end{pmatrix} = f, \ x \in \Omega, \\ \widetilde{K}(p,x)^{-1}Z(p,x)\begin{pmatrix} e \\ h \end{pmatrix} + \mathbb{Q}\begin{pmatrix} e \\ h \end{pmatrix} \\ \qquad\qquad + \widetilde{K}(p,x)^{-1}\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \widetilde{K}(p,x)^{-1}\mathbb{Q}f. \end{cases} \tag{3.6}$$

As $\det(K(p,x)) = \det(\widetilde{K}(p,x))^3$, $\widetilde{K}(p,x)$ is invertible thanks to (B2) and the above equation makes sense. Now introduce the closed unbounded operator $\mathbb{T}$

$$\mathbb{T} = \begin{pmatrix} 0 & -\nabla\times & \nabla & 0 \\ \nabla\times & 0 & 0 & \nabla \\ \mathrm{div} & 0 & 0 & 0 \\ 0 & \mathrm{div} & 0 & 0 \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathbb{T}) = \left\{ U \in (H(\mathrm{curl},\Omega) \cap H(\mathrm{div},\Omega))^2 \times H^1(\Omega)^2 \mid U(x)_{|\partial\Omega} \in \ker(\widetilde{N}(x)) \right\}$$

for $U = (e,h,\varphi,\psi)$ and where the boundary conditions are encoded by the linear application $\widetilde{N} : \partial\Omega \longrightarrow \mathrm{Hom}(\mathbb{C}^8, \mathbb{C}^5))$ which is Lipschitz continuous and defined by

$$\widetilde{N}(x)U(x) = \begin{pmatrix} \nu \times e(x) + \nu \times \Lambda(x)(\nu \times h(x)) \\ \varphi(x) \\ \psi(x) \end{pmatrix} \text{ for } x \in \partial\Omega.$$

The elliptized Maxwell's equations (3.6) can now be summarized as follows:

$$\textit{Find } U = (e,h,\varphi,\psi) \in \mathcal{H}_{\mathbb{T}} \textit{ such that}$$
$$\begin{cases} (\widetilde{K}(p,x) + \mathbb{T})U(p,x) = F(p,x), \ x \in \Omega, \\ U(p,x) \in \ker(\widetilde{N}(x)), \ x \in \partial\Omega, \end{cases} \tag{3.7}$$

where $\widetilde{K}(p,.)$ belongs to $L^\infty(\overline{\Omega}, \mathrm{Hom}(\mathbb{C}^8))$ according to (B1) and Rademacher's theorem. Furthermore $F(p,.) := (f, \widetilde{K}(p,.)^{-1}\mathbb{Q}f) \in L^2(\Omega)^8$ is holomorphic for all $p \in D_0$.

LEMMA 3.3. *Assume that* $F(p,.) = (f, \widetilde{K}(p,.)^{-1}\mathbb{Q}f)$ *for* $f \in (H(\mathrm{div}, \Omega))^2$ *and* $U = (e, h, \varphi, \psi) \in \mathcal{D}(\mathbb{T})$ *satisfies* (3.7). *Then* $\varphi = \psi = 0$ *and* $u = (e, h) \in \mathcal{D}(\mathbb{M})$ *is a solution to* (3.4).

From Lemma 3.3, (3.4) corresponds to solving (3.7). Moreover, we have the next lemma.

LEMMA 3.4. $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$ *is maximal dissipative with compact resolvent.*

From Lemma 3.4 and as the multiplicative operator $\widetilde{K}(p,.)$ is bounded for all p in D_0, the resolvent set of $(\widetilde{K}(p,.) + \mathbb{T}, \mathcal{D}(\mathbb{T}))$ is thus nonempty for all $p \in D_0$. Moreover, the holomorphic family of closed operators $(\widetilde{K}(p,.) + \mathbb{T}, \mathcal{D}(\mathbb{T}))_{p\in D_0}$ has compact resolvent. Hence, solving (3.6) is the same as inverting a holomorphic family of closed operators with compact resolvent. This can actually be done with Fredholm analytical theory [19, Theorem 1.10, p. 371] since we find $p_0 \in D_0$ such that (3.6) is well-posed. This is the purpose of the lemma below.

LEMMA 3.5. *The operator* $(\widetilde{K}(p_0,.)+\mathbb{T}, \mathcal{D}(\mathbb{T}))$ *for* p_0 *satisfying* (B3) *(respectively,* (C3)*) is invertible and* $(\widetilde{K}(p_0,.) + \mathbb{T})^{-1} \in \mathcal{B}(L^2(\Omega)^8, \mathcal{D}(\mathbb{T}))$.

The notation $\mathcal{B}(\mathbf{X}, \mathbf{Y})$ stands for the set of bounded linear operators from $\mathbf{X}$ to $\mathbf{Y}$, both being Banach spaces. Using Lemma 3.5 and Fredholm analytical theory, we obtain that the holomorphic family of closed operators with compact resolvent $(\widetilde{K}(p,.) + \mathbb{T}, \mathcal{D}(\mathbb{T}))_{p\in D_0}$ is invertible for $p \in D_0\backslash S$, where S is a discrete, locally finite, and possibly empty set of D_0. This shows that the system (3.7) is well-posed for $p \in D_0\backslash S$. Furthermore, using (B1) and (B2) (respectively, (C1) and (C2)), the applications $p \in D_0 \longmapsto (\widetilde{K}(p,.) + \mathbb{T}) \in \mathcal{B}(\mathcal{D}(\mathbb{T}), L^2(\Omega)^8)$ and $p \in D_0 \longmapsto F(p) \in L^2(\Omega)^8$ are both holomorphic and so is (see [19, p. 365]) the application: $U : p \in D_0\backslash S \longmapsto U(p,.) := (\widetilde{K}(p,.) + \mathbb{T})^{-1}F(p,.) \in L^2(\Omega)^8$. Consequently the application solution to (3.7) $p \in D_0\backslash S \longmapsto u(p,.) \in L^2(\Omega)^6$, where $u(p,.)$ satisfies (3.4), is holomorphic. Moreover we get, for some constant $C(p) > 0$, the estimate

$$\|U(p,.)\|_{L^2(\Omega)^8} \le C(p) \|F(p,.)\|_{L^2(\Omega)} .$$

Now using Lemma 3.3 we obtain that the previous estimate reduces to

$$\|e(p,.)\|_{L^2(\Omega)^3} + \|h(p,.)\|_{L^2(\Omega)^3} \le C(p) \left\{ \|f\|_{L^2(\Omega)} + \|\mathbb{Q}f\|_{L^2(\Omega)} \right\},$$

which concludes the proof. □

The rest of the section is dedicated to the proof of the lemmas set up above.

Proof of Lemma 3.3. We follow the proof of Theorem 4.2 in [22]. Applying $\mathbb{Q}$ to the first equation of (3.6) shows that

$$\begin{pmatrix} \Delta\varphi \\ \Delta\psi \end{pmatrix} = \mathbb{Q}f - \mathbb{Q}\left(K(p,.) \begin{pmatrix} e \\ h \end{pmatrix} \right),$$

where $\Delta\psi = \mathrm{div}\,(\nabla\psi)$. From the definition of $\widetilde{K}(p, x)$ and $Z(p, x)$, we derive that the second equation of (3.6) can be written as follows:

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \mathbb{Q}f - \mathbb{Q}\left(K(p,.) \begin{pmatrix} e \\ h \end{pmatrix} \right).$$

As a result, φ and ψ are both solutions to the following equation:

$$\begin{array}{c} \textit{Find } \vartheta \in H^1_0(\Omega) \textit{ such that} \\ \Delta\vartheta - \vartheta = 0 \text{ on } \Omega \end{array}$$

for $\vartheta \in \{\varphi, \psi\}$, which only admits the null solution. □

Proof of Lemma 3.4. There exist three symmetric tensors $T_j \in \mathrm{Hom}(\mathbb{C}^8)$ such that $\mathbb{T} = \sum_{j=1}^{3} T_j \partial_j$. Thus, according to [31], the unbounded operator $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$ is maximal dissipative if

(a) on every connected component of $\partial\Omega$, the rank of $\mathbb{T}_\nu(x) := \sum_{j=1}^{3} T_j \nu_j(x)$, $x \in \partial\Omega$ is constant;

(b) for all $x \in \partial\Omega$ and $U(p,x) \in \ker(\widetilde{N}(x))$, we have $\langle \mathbb{T}_\nu(x) U(p,x), \overline{U(p,x)} \rangle \geq 0$;

(c) $\dim(\ker(\widetilde{N}(x))= \sharp\{$nonnegative eigenvalues of $\mathbb{T}_\nu$ counting multiplicity$\}$.

First, straightforward computations shows that $\det(\mathbb{T}_\nu) = (\nu_1^2 + \nu_2^2 + \nu_3^2)^4 > 0$ and then (a) is satisfied. Then, for $U(x) = (e, h, \varphi, \psi) \in \ker(\widetilde{N}(x))$,

$$\left\langle \mathbb{T}_\nu(x) U(x), \overline{U(x)} \right\rangle = 2\mathcal{R}e \left\langle \Lambda(x)(\nu \times h), \overline{\nu \times h} \right\rangle \geq 2\alpha |\nu \times h|^2 \geq 0,$$

verifying (b). Finally, the spectrum of $\mathbb{T}_\nu$ is $\sigma(\mathbb{T}_\nu) = \{-1, +1\}$ both with multiplicity 4 and $\dim(\ker(\widetilde{N}(x))) = 4$, so (c) is satisfied too.

Now, from [24], the following inequality holds for all $u \in \mathcal{D}(\mathbb{M}) \cap (H(\mathrm{div}, \Omega))^2$:

$$\|u\|_{(H^1(\Omega))^6} \leq C \left\{ \|u\|_{(L^2(\Omega))^6} + \|\mathbb{M}u\|_{(L^2(\Omega))^6} + \|\mathbb{Q}u\|_{(L^2(\Omega))^6} \right\}.$$

Thus, the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ being compact, one obtains that the embedding of $\mathcal{D}(\mathbb{T})$ into $L^2(\Omega)^8$ is compact. Hence, the resolvent of $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$ is compact. ◻

Proof of Lemma 3.5. Inverting $(\widetilde{K}(p_0, .) + \mathbb{T})$ on $\mathcal{D}(\mathbb{T})$ means solving the following problem:

$$\text{(3.8)} \qquad \begin{array}{l} \textit{Find } U = (e, h, \varphi, \psi) \in \mathcal{H}_{\mathbb{T}} \textit{ such that} \\ \left\{ \begin{array}{l} (\widetilde{K}(p_0, x) + \mathbb{T}) U(p, x) = G, \ x \in \Omega, \\ U(p, x) \in \ker(\widetilde{N}(x)), \ x \in \partial\Omega, \end{array} \right. \end{array}$$

for G in $L^2(\Omega)^8$. From Lemma 3.4, there exists at least one $\alpha > 0$ belonging to the resolvent set of the operator $(\mathbb{T}, \mathcal{D}(\mathbb{T}))$. Furthermore the resolvent operator $(\alpha \mathbb{I}_8 + \mathbb{T})^{-1}$ is a compact operator of $L^2(\Omega)^8$. Then (3.8) remains to solve the following problem:

$$\text{(3.9)} \qquad \begin{array}{l} \textit{Find } U \in L^2(\Omega)^8 \textit{ such that} \\ \left(\mathbb{I}_8 + (\alpha \mathbb{I}_8 + \mathbb{T})^{-1} \left(\widetilde{K}(p_0, .) - \alpha \mathbb{I}_8 \right) \right) U = (\alpha \mathbb{I}_8 + \mathbb{T})^{-1} G \quad =: \widetilde{G}. \end{array}$$

The boundedness of $\widetilde{K}(p_0, x)$ shows that the operator $(\alpha \mathbb{I}_8 + \mathbb{T})^{-1} (\widetilde{K}(p_0, .) - \alpha \mathbb{I}_8)$ is compact on $L^2(\Omega)^8$. Then, from the Fredholm alternative, we only need to show the injectivity of the operator $(\mathbb{I}_8 + (\alpha \mathbb{I}_8 + \mathbb{T})^{-1} (\widetilde{K}(p_0, .) - \alpha \mathbb{I}_8))$ acting on $L^2(\Omega)^8$ to prove the lemma. Taking $\widetilde{G} = 0$ in (3.9) and using the boundedness of $(\alpha \mathbb{I}_8 + \mathbb{T})^{-1} : L^2(\Omega)^8 \longmapsto \mathcal{D}(\mathbb{T})$ implies that U belongs to $\mathcal{D}(\mathbb{T})$ and satisfies (3.8) with right member $G = 0$. Now Lemma 3.3 shows that $U = (e, h, \varphi, \psi) = (e, h, 0, 0)$ with (e, h) verifying (3.4) for $p = p_0$. Then from (B3) (respectively, (C3)) and the uniqueness of the solution to (3.4) we infer that $(e, h) = (0, 0)$, concluding the proof of the lemma. ◻

3.2. Generic well-posedness for some anisotropic materials. The generic well-posedness result for the Maxwell systems (3.1), (3.3), or (3.2) proved in the previous section can be applied only to materials with scalar physical parameters. In the case of anisotropic materials, we would have to take into account some physical parameters that are no longer scalar. This is done in this section. We start by introducing the corresponding assumptions.

Assumption 5 (for bi-anisotropic materials having tensorial parameters).

(BT1) The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$, and $\zeta(p,x)$ (which are now 3×3 tensors) are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(BT2) The applications $\varepsilon(p,x), \mu(p,x), \xi(p,x)$, and $\zeta(p,x)$ belong to $L^\infty(\overline{\Omega})$ for all $p \in D_0$. Moreover, $\det(\varepsilon(p,x)\mu(p,x) - \xi(p,x)\zeta(p,x)) \neq 0$ for almost all $x \in \overline{\Omega}$ and for any $p \in D_0$.

(BT3) There exist $p_0 \in D_0$ and $\alpha > 0$ such that the following inequality holds:

$$\begin{aligned}&\mathcal{R}e\left\{\langle p_0\varepsilon(p_0,x)X,\overline{X}\rangle + \langle p_0\mu(p_0,x)Y,\overline{Y}\rangle + \langle p_0\xi(p_0,x)Y,\overline{X}\rangle\right\}\\ &\quad + \mathcal{R}e\left\{\langle p_0\zeta(p_0,x)X,\overline{Y}\rangle\right\} \geq \alpha(|X|^2+|Y|^2)\end{aligned}$$

for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

(BT4) There exists $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, verifying

$$\begin{aligned}&\mathcal{R}e\left\{\langle p\varepsilon(p,x)a(p,x)X,\overline{X}\rangle + \langle p\mu(p,x)a(p,x)Y,\overline{Y}\rangle\right\}\\ &\quad + \mathcal{R}e\left\{\langle p\xi(p,x)a(p,x)Y,\overline{X}\rangle + \langle p\zeta(p,x)a(p,x)X,\overline{Y}\rangle\right\}\\ &\geq \alpha(|X|^2+|Y|^2)\end{aligned}$$

for all $p \in D_0$, for almost all $x \in \Omega$, and for all $X, Y \in \mathbb{C}^3$.

Assumption 6 (for dielectric materials having tensorial parameters). Suppose that (BT1), (BT2), (BT3) are holding and replace (BT4) with the following:

(DT4) Parameters ξ and ζ are vanishing, and there exists $a_\varepsilon(p,x),\ a_\mu(p,x) \in \mathbb{C}$ both Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$. Moreover, there exists $\alpha > 0$ such that the following inequality holds:

$$\mathcal{R}e\left\{\langle p\vartheta(p,x)a_\vartheta(p,x)X,\overline{X}\rangle\right\} \geq \alpha|X|^2$$

for $\vartheta \in \{\varepsilon,\mu\}$, for all $p \in D_0$, for almost all $x \in \Omega$, and for all $X, Y \in \mathbb{C}^3$.

Assumption 7 (for chiral materials having tensorial parameters).

(CT1) The applications $\varepsilon(p,x), \mu(p,x)$, and $\beta(p,x)$ (which are now 3×3 tensors) are holomorphic in $p \in D_0$ for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.

(CT2) The applications $\varepsilon(p,x), \mu(p,x)$, and $\beta(p,x)$ are in $L^\infty(\overline{\Omega})$ for all $p \in D_0$. Moreover, we ask for

$$\det\left[p^2\varepsilon(p,x)\mu(p,x)\right]\det\widetilde{\mathcal{M}} \neq 0$$

for almost all $x \in \overline{\Omega}$ with $\widetilde{\mathcal{M}}(p,x) = \left(\mathbb{I}_3 + p^2\beta(p,x)\varepsilon(p,x)\beta(p,x)\mu(p,x)\right)^{-1}$.

(CT3) There exist p_0 in D_0 and $\alpha > 0$ such that the inequality

$$\begin{aligned}&\mathcal{R}e\left\{\langle p_0\widetilde{\mathcal{M}}(p_0,x)\varepsilon(p_0,x)X,\overline{X}\rangle + \langle p_0\mathcal{M}(p_0,x)\mu(p_0,x)Y,\overline{Y}\rangle\right\}\\ &\quad + \mathcal{R}e\left\{\langle p_0^2\beta(p_0,x)\varepsilon(p_0,x)\mathcal{M}(p_0,x)\mu(p_0,x)Y,\overline{X}\rangle\right\}\\ &\quad - \mathcal{R}e\left\{\langle p_0^2\widetilde{\mathcal{M}}(p_0,x)\beta(p_0,x)\mu(p_0,x)\varepsilon(p_0,x)X,\overline{Y}\rangle\right\} \geq \alpha(|X|^2+|Y|^2),\end{aligned}$$

where $\mathcal{M}(p,x) = (\mathbb{I}_3 + p^2\beta(p,x)\mu(p,x)\beta(p,x)\varepsilon(p,x))^{-1}$, holds for almost all $x \in \Omega$ and for all $X, Y \in \mathbb{C}^3$.

(CT4) There exists $a(p,x) \in \mathbb{C}$ Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$ such that the following inequality holds:

$$\begin{aligned}&\mathcal{R}e\left\{\langle pa(p,x)\widetilde{\mathcal{M}}(p,x)\varepsilon(p,x)X,\overline{X}\rangle + \langle pa(p,x)\mathcal{M}(p,x)\mu(p,x)Y,\overline{Y}\rangle\right\}\\ &+\mathcal{R}e\left\{\langle p^2a(p,x)\beta(p,x)\varepsilon(p,x)\mathcal{M}(p,x)\mu(p,x)Y,\overline{X}\rangle\right\}\\ &-\mathcal{R}e\left\{\langle p^2a(p,x)\widetilde{\mathcal{M}}(p,x)\beta(p,x)\mu(p,x)\varepsilon(p,x)X,\overline{Y}\rangle\right\}\geq\alpha(|X|^2+|Y|^2)\end{aligned}$$

for all $p \in D_0$, for almost all $x \in \Omega$, and for all $X, Y \in \mathbb{C}^3$.

We can now formulate the following theorem.

Theorem 3.6. *If Assumption* 5 *(respectively,* 6 *or* 7*) is satisfied, then for all* $(j,m) \in L^2(\Omega)^3 \times L^2(\Omega)^3$ *Maxwell's system* (3.2) *(respectively,* (3.1) *or* (3.3)*) has a unique solution for all* p *in* $D_0\backslash S$*, where* $S \subset D_0$ *is a discrete, locally finite, and possibly empty set of* D_0*. Moreover the solution satisfies the bound*

$$\forall p \in D_0\backslash S, \|(e(p,.),h(p,.))\|_{L^2(\Omega)} \leq C(p)\,\|(j,m)\|_{L^2(\Omega)}\,,$$

and the application $p \in D_0\backslash S \longmapsto (e(p,.),h(p,.)) \in L^2(\Omega)^6$ *is holomorphic.*

Proof. We focus on system (3.4) when the multiplicative operator $K(p,x)$ satisfies (BT1), (BT2), (BT3) and either (BT4) or (DT4). The case of media checking (CT1), (CT2), (CT3), (CT4) is very similar.

Let us introduce the Hodge decomposition [9]

$$L^2(\Omega)^3 \times L^2(\Omega)^3 = (H(\mathrm{div}0,\Omega))^2 \oplus \mathrm{grad}(H_0^1(\Omega))^2, \tag{3.10}$$

where $H(\mathrm{div}0,\Omega) = \{v \in L^2(\Omega)^3 \mid \mathrm{div}v = 0\}$, and denote by $P_0 : L^2(\Omega)^6 \longrightarrow H(\mathrm{div}0,\Omega)^2$ and $P_\nabla : L^2(\Omega)^6 \longrightarrow \mathrm{grad}(H_0^1(\Omega))^2$ the orthogonal projections associated to the direct sum (3.10). Applying them to (3.4) and using the identity $P_\nabla\mathbb{M}u = 0$ for all $u \in \mathcal{D}(\mathbb{M})$, the solution to (3.4) thus solves the system

$$\begin{array}{l}Find\ u = P_0u + P_\nabla u \in \mathcal{D}(\mathbb{M})\ such\ that\\ \left\{\begin{array}{l}P_0K(p,.)(P_0u+P_\nabla u) + P_0\mathbb{M}P_0u = P_0f,\\ P_\nabla K(p,.)(P_\nabla u + P_0u) = P_\nabla f.\end{array}\right.\end{array} \tag{3.11}$$

The second equation of (3.11) is solved with the following lemma.

Lemma 3.7. *Assume that* (B1), (B2), (B3) *and either* (BT4) *or* (DT4) *hold. Then, the operator* $P_\nabla K(p,.)P_\nabla \in \mathcal{B}(\mathrm{grad}(H_0^1(\Omega))^2)$ *is invertible with a bounded inverse for all* $p \in D_0\backslash S_0$*, where* S_0 *is a discrete, locally finite, and possibly empty set of* D_0*. Moreover the application* $p \in D_0\backslash S_0 \longmapsto (P_\nabla K(p,.)P_\nabla)^{-1} \in \mathcal{B}(\mathrm{grad}(H_0^1(\Omega))^2)$ *is holomorphic.*

Applying Lemma 3.7, the system (3.11) becomes

$$\begin{array}{l}Find\ u = P_0u + P_\nabla u \in \mathcal{D}(\mathbb{M})\ such\ that\\ \left\{\begin{array}{l}P_\nabla u = (P_\nabla K(p,.)P_\nabla)^{-1}\left[P_\nabla f - P_\nabla K(p,.)P_0u\right],\\ B(p,.)P_0u + P_0\mathbb{M}P_0u = \widetilde{f}(p),\end{array}\right.\end{array} \tag{3.12}$$

where $\widetilde{f}(p,.) = P_0f - P_0K(p,.)P_\nabla\left(P_\nabla K(p,.)P_\nabla\right)^{-1}P_\nabla f$ belongs to $H(\mathrm{div}0,\Omega)^2$, and

$$B(p,.) = P_0K(p,.)P_0 - P_0K(p,.)P_\nabla(P_\nabla K(p,.)P_\nabla)^{-1}P_\nabla K(p,.)P_0 \in \mathcal{B}\left(H(\mathrm{div}0,\Omega)^2\right).$$

Let $\widetilde{\mathbb{M}}$ be the restriction of the operator $\mathbb{M}$ to the set $(H(\mathrm{div}0,\Omega))^2$. Then (3.12) is equivalent to the inversion of the holomorphic family of the closed operator of

$(H(\mathrm{div}0,\Omega))^2$ defined by $(B(p,.)+\widetilde{\mathbb{M}},(H(\mathrm{div}0,\Omega))^2\cap\mathcal{D}(\mathbb{M}))_{p\in D_0\backslash S_0}$. Consider now the following Majda inequality [24] holding for all $u\in\mathcal{D}(\mathbb{M})\cap(H(\mathrm{div},\Omega))^2$:

$$\|u\|_{(H^1(\Omega))^6}\leq C\left\{\|u\|_{(L^2(\Omega))^6}+\|\mathbb{M}u\|_{(L^2(\Omega))^6}+\|\mathbb{Q}u\|_{(L^2(\Omega))^6}\right\}, \tag{3.13}$$

where $\mathbb{Q}$ is defined in Theorem 3.2. From (3.13), the compactness of the embedding of $H^1(\Omega)$ into $L^2(\Omega)$, and the maximal dissipativity of $(\mathbb{M},\mathcal{D}(\mathbb{M}))$, one obtains that the holomorphic family of closed operators $(B(p,.)+\widetilde{\mathbb{M}},(H(\mathrm{div}0,\Omega))^2\cap\mathcal{D}(\mathbb{M}))_p$ has a compact resolvent and nonempty resolvent set for all $p\in D_0\backslash S_0$. From (BT3), there exists p_0 such that $K(p_0,x)$ is coercive for almost all $x\in\Omega$. Then, since the unbounded operator $(\mathbb{M},\mathcal{D}(\mathbb{M}))$ is maximal dissipative, (3.4) is well-posed for $p=p_0$. Hence, by the equivalence between (3.4) and (3.12), $B(p_0,.)+\widetilde{\mathbb{M}}$ is invertible on $(H(\mathrm{div}0,\Omega))^2\cap\mathcal{D}(\mathbb{M})$. Finally, thanks to the Fredholm analytic theory $B(p,.)+\widetilde{\mathbb{M}}$ is invertible on $(H(\mathrm{div}0,\Omega))^2\cap\mathcal{D}(\mathbb{M})$ for all $p\in D_0\backslash S$, where S_1 is a discrete, locally finite, and possibly empty set of $D_0\setminus S_0$.

Note that the inverse operator $(B(p,.)+\widetilde{\mathbb{M}})^{-1}$ belongs to $\mathcal{B}(H(\mathrm{div}0,\Omega)^2)$ and that the application $p\in D_0\backslash S\longmapsto(B(p,.)+\widetilde{\mathbb{M}})^{-1}$ is holomorphic. Thus $P_0u(p,.)$ is holomorphic too. Then, from Lemma 3.7 we derive that $P_\nabla u(p,.)$ is holomorphic on $D_0\backslash S$ and so is the application $p\in D_0\backslash S\longmapsto u(p,.)=P_\nabla u(p,.)+P_0u(p,.)\in L^2(\Omega)^6$. Finally we have the bound

$$\|u(p,.)\|_{L^2(\Omega)}\leq\|P_\nabla u(p,.)\|_{L^2(\Omega)}+\|P_0u(p,.)\|_{L^2(\Omega)}\leq C(p)\,\|f\|_{L^2(\Omega)},$$

where $C(p)$ is a positive constant depending only on Ω and p. □

The rest of this section is dedicated to proving the lemmas.

Proof of Lemma 3.7. We begin by proving the lemma when (BT1), (BT2), (BT3), (BT4) hold. Inverting $P_\nabla K(p,.)P_\nabla$ on $\mathrm{grad}(H_0^1(\Omega))^2$ summarizes as solving the problem

$$\begin{array}{c}\textit{Find }\varphi\in H_0^1(\Omega)^2\textit{ such that for all }v\in H_0^1(\Omega)^2\\ \displaystyle\int_\Omega\left\langle K(p,x)\mathbb{G}\varphi,\overline{\mathbb{G}v}\right\rangle dx=\langle h,v\rangle_{H^{-1}(\Omega)^2\times H_0^1(\Omega)^2},\end{array} \tag{3.14}$$

where h belongs to $H^{-1}(\Omega)^2$ and $\mathbb{G}\varphi=(-\nabla\varphi_1,-\nabla\varphi_2)$ for $\varphi=(\varphi_1,\varphi_2)\in H_0^1(\Omega)^2$. System (3.14) is equivalent to the following second order partial differential equation:

$$\begin{array}{c}\textit{Find }\varphi\in H_0^1(\Omega)^2\textit{ such that}\\ -\mathbb{Q}\left(K(p,.)\mathbb{G}\varphi(p,x)\right)=h.\end{array} \tag{3.15}$$

Unfortunately, the principal part of this second order partial differential operator fails to be coercive for some $p\in D_0$. However, we introduce the following change of unknown:

$$\varphi(p,.)=a(p,.)\psi(p,.), \tag{3.16}$$

where a is given by (BT4). Putting (3.16) into (3.15) leads to an equivalent in ψ:

$$\begin{array}{l}\textit{Find }\psi\in H_0^1(\Omega)^2\textit{ such that}\\ \mathbb{S}(p)\psi:=-\mathbb{Q}K(p,x)\left(a(p,x)\mathbb{G}\psi(p,x)+\begin{pmatrix}\psi_1(p,x)\nabla a(p,x)\\ \psi_2(p,x)\nabla a(p,x)\end{pmatrix}\right)=h.\end{array} \tag{3.17}$$

From (BT4) the multiplicative operator $a(p,.)K(p,.)$ is coercive for all $p\in D_0$, implying that the principal part of (3.17) is coercive. Moreover, $a(p,.)$ is Lipschitz continuous for all $p \in D_0$ so Rademacher's theorem shows that $\nabla a(p,.)$ belongs to $L^\infty(\Omega)$. Thus, $\mathbb{S}(p) : H_0^1(\Omega)^2 \to H^{-1}(\Omega)^2$ is a holomorphic family of closed operators.

Let us assume now that there exists λ in the resolvent set of $(\mathbb{S}(p), H_0^1(\Omega)^2)$. Since the embedding of $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ is compact, the resolvent operator $(\mathbb{S}(p) - \lambda)^{-1}$ defines a compact operator of $H^{-1}(\Omega)^2$. Consequently, solving (3.17) is the same as inverting a holomorphic family of closed operators with compact resolvent. This is achieved with help of the Fredholm analytic theory since we show that the resolvent set of $(\mathbb{S}(p), H_0^1(\Omega)^2)$ is nonempty for all $p \in D_0$. Thus, let λ be an arbitrary complex number and $A_p(\psi, v)$ be given, for any $\psi, v \in H_0^1(\Omega)^2$, by

$$A_p(\psi, v) = \int_\Omega \left\langle K(p,x)a(p,x)\mathbb{G}\psi, \overline{\mathbb{G}v}\right\rangle + \left\langle K(p,x)\begin{pmatrix} \psi_1(p,x)\nabla a(p,x) \\ \psi_2(p,x)\nabla a(p,x)\end{pmatrix}, \overline{\mathbb{G}v}\right\rangle dx.$$

Hypothesis (BT4) together with the Cauchy–Schwarz inequality imply then the bound

$$\begin{aligned}\mathcal{R}e(A_p(\psi,\psi)) + \mathcal{R}e(\lambda)\int_\Omega |\psi|^2 dx \geq \alpha \parallel \mathbb{G}\psi \parallel^2_{L^2(\Omega)} + \mathcal{R}e(\lambda)\left\|\psi\right\|^2_{L^2(\Omega)} \\ - 2\left\|K(p,.)\mathbb{G}a(p,.)\right\|_{L^\infty(\Omega)}\left\|\psi\right\|_{L^2(\Omega)^2}\left\|\mathbb{G}\psi\right\|_{L^2(\Omega)^2}.\end{aligned}$$

Using Young's inequality $ab \leq \xi a^2/2 + b^2/(2\xi)$ in the term $\|\psi\|_{L^2(\Omega)^2}\|\mathbb{G}\psi\|_{L^2(\Omega)^2}$ with $\xi = \alpha/(2\|K(p,.)\mathbb{G}a(p,.)\|_{L^\infty(\Omega)})$ and gathering the previous calculations yields

$$\begin{aligned}\mathcal{R}e(A_p(\psi,\psi)) + \mathcal{R}e(\lambda)\int_\Omega |\psi|^2 dx \geq \left(\mathcal{R}e(\lambda) - \frac{\left\|K(p,.)\mathbb{G}a(p,.)\right\|^2_{L^\infty(\Omega)}}{\alpha}\right)\left\|\psi\right\|^2_{L^2(\Omega)^2} \\ + \frac{\alpha}{2}\left\|\mathbb{G}\psi\right\|^2_{L^2(\Omega)}.\end{aligned}$$

Thus, taking $\lambda \in \mathbb{C}$ with a big enough real part and using the Lax–Milgram theorem, we obtain that the resolvent set of $(\mathbb{S}(p), H_0^1(\Omega)^2)$ is nonempty for all $p \in D_0$. It remains to find p such that (3.17) is well-posed. Taking $p = p_0$ defined through assumption (BT3) we get that (3.14) is well-posed and the reverse change of unknown (3.16) implies that (3.17) is well-posed too for $p = p_0$. Hence the holomorphic family of closed operator $\left(\mathbb{S}(p), H_0^1(\Omega)^2\right)$ is invertible for all $p \in D_0 \backslash S_0$, where S_0 is a discrete, locally finite, and possibly empty set of D_0. Moreover, the inverse operator $\mathbb{S}(p)^{-1}$ which belongs to $\mathcal{B}(H^{-1}(\Omega)^2, H_0^1(\Omega)^2)$ is holomorphic for all $p \in D_0 \backslash S_0$ because the application $p \in D_0 \longmapsto \mathbb{S}(p) \in \mathcal{B}(H_0^1(\Omega)^2, H^{-1}(\Omega)^2)$ is holomorphic [19, p. 365].

Since (3.17) and (3.14) are equivalent, we derive from the well-posedness of (3.17) the invertibility of the operator $P_\nabla K(p,.)P_\nabla$ on $(\mathrm{grad}(H_0^1(\Omega)))^2$ for all $p \in D_0 \backslash S_0$. Moreover, the closed graph theorem shows that $(P_\nabla K(p,.)P_\nabla)^{-1}$ actually belongs to $\mathcal{B}(\mathrm{grad}(H_0^1(\Omega))^2)$. Finally, note that the operator $(P_\nabla K(p,.)P_\nabla)^{-1} \in \mathcal{B}(\mathrm{grad}(H_0^1(\Omega))^2)$ is holomorphic for all $p \in D_0 \backslash S_0$ since the application $p \in D_0 \longmapsto P_\nabla K(p,.)P_\nabla \in \mathcal{B}(\mathrm{grad}(H_0^1(\Omega))^2)$ is holomorphic [19, p. 361].

When (DT4) holds instead of (BT4), the tensor $K(p,x)$ is the block-diagonal matrix $K(p,x) = p\,\mathrm{diag}(\varepsilon(p,x), \mu(p,x))$. Hence, (3.14) reduces to

$$\begin{aligned}&\textit{Find } \varphi \in H_0^1(\Omega) \textit{ such that} \\ &-\mathrm{div}\,(p\vartheta(p,x)\nabla\varphi_\vartheta) = h_\vartheta\end{aligned}$$

for $h_\vartheta \in H^{-1}(\Omega)$ and $\vartheta \in \{\varepsilon, \mu\}$. Introducing the changes of unknown $\varphi_\vartheta(p,.) = a_\vartheta(p,.)\psi_\vartheta(p,.)$, where a_ϑ is given in (DT4), yields to

$$\begin{aligned}&\textit{Find } \psi \in H_0^1(\Omega) \textit{ such that} \\ &\quad -\mathrm{div}\,(\vartheta(p,x)a_\vartheta(p,x)\nabla\psi_\vartheta(p,x)) - \mathrm{div}\,(\vartheta(p,x)\psi_\vartheta(p,x)\nabla a_\vartheta(p,x)) = h_\vartheta \text{ on } \Omega.\end{aligned}$$

Now, the proof of the existence and uniqueness of ψ_ϑ can be achieved in the same way as before using the Fredholm analytic theory once for each $\vartheta \in \{\varepsilon, \mu\}$. ∎

4. Acoustic wave system. In this section we investigate the first order wave equation

$$\text{(4.1)} \qquad \begin{array}{l} \textit{Find } (u,\rho) \in H(\mathrm{div},\Omega) \times H^1(\Omega) \textit{ such that} \\ \left\{ \begin{array}{l} \Gamma(p,x)u(p,x) - \nabla\rho(p,x) = f_1(x) \text{ on } \Omega, \\ n(p,x)\rho(p,x) - \mathrm{div}(u(p,x)) = f_2(x) \text{ on } \Omega, \\ \rho(p,x) + \lambda(x)\langle u(p,x), \nu\rangle = 0 \text{ on } \partial\Omega, \end{array} \right. \end{array}$$

where ρ denotes the acoustic velocity, u is the acoustic pressure, $f = (f_1, f_2) \in L^2(\Omega)^3 \times L^2(\Omega)$ is a source term, λ is the acoustic impedance, Γ is the bulk modulus, and n is the refractive index. The normal trace of $u \in H(\mathrm{div},\Omega)$ appearing in the boundary conditions has to be understood in the sense that $\langle u, \nu\rangle_{|\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$. Finally, assume that the function $\lambda : \partial\Omega \mapsto \mathbb{C}$ is Lipschitz continuous with $\mathcal{R}e(\lambda) \geq 0$.

We introduce the following conditions to be satisfied by the material.

Assumption 8 (for acoustic materials).

- (A1) The applications $\Gamma(p,x)$ and $n(p,x)$ are holomorphic in p on D_0 for almost all $x \in \Omega$, where D_0 is a connected open set of $\mathbb{C}$.
- (A2) The applications $\Gamma(p,x)$ and $n(p,x)$ both belong to $L^\infty(\overline{\Omega})$ and are invertible for all $p \in D_0$.
- (A3) There exists p_0 in D_0 and $\alpha > 0$ such that the following inequality holds:

$$\mathcal{R}e\left\{\langle \Gamma(p_0,x)X, \overline{X}\rangle\right\} + |z|^2\mathcal{R}e(n(p_0,x)) \geq \alpha(|X|^2 + |z|^2)$$

for all $(X,z) \in \mathbb{C}^3 \times \mathbb{C}$ and for almost all $x \in \Omega$.

Our first generic well-posedness result for (4.2) is given next.

THEOREM 4.1. *Suppose that $\Gamma(p) \in \mathbb{C}\backslash\{0\}$ is scalar valued and does not depend on x, and assume that Assumption* 8 *is fulfilled. Then* (4.2) *is well-posed for all $p \in D_0\backslash S$, where $S \subset D_0$ is a discrete, locally finite, and possibly empty set of D_0. Moreover, the solution is holomorphic from $D_0\backslash S$ to $L^2(\Omega)^4$ and continuous with respect to the data.*

Proof. First, rewrite (4.1) as the following general first order wave equation:

$$\text{(4.2)} \qquad \begin{array}{l} \textit{Find } (u,\rho) \in \mathcal{H}_{\mathbb{W}} \textit{ such that} \\ \left\{ \begin{array}{l} (K(p,x) - \mathbb{W})\begin{pmatrix} u \\ \rho \end{pmatrix} = f(x), \ x \in \Omega, \\ \rho(p,x) + \lambda(x)\langle u(p,x), \nu\rangle = 0, \ x \in \partial\Omega, \end{array} \right. \end{array}$$

where $f \in L^2(\Omega)^4$ and $K(p,x) = \begin{pmatrix} \Gamma(p) & 0 \\ 0\mathbb{I}_3 & n(p,x) \end{pmatrix}$. The operator $\mathbb{W}$ is defined by

$$\mathbb{W} = \begin{pmatrix} 0_{3\times 3} & \nabla \\ \mathrm{div} & 0 \end{pmatrix},$$

whose domain is

$$\mathcal{D}(\mathbb{W}) = \left\{(u,\rho) \in H(\mathrm{div},\Omega) \times H^1(\Omega) \ |\rho(x) + \lambda(x)\langle u(x), \nu\rangle = 0, \ x \in \partial\Omega\right\}.$$

The operator $(-\mathbb{W}, \mathcal{D}(\mathbb{W}))$ is maximal dissipative [31]. Now consider the Hodge decomposition [4, Theorem 10, p. 54]

$$\text{(4.3)} \qquad L^2(\Omega)^3 = \mathrm{grad}H^1(\Omega) \oplus \left(\nabla \times \widetilde{V}\right),$$

where $\widetilde{V} = \{\Psi \in H(\text{curl}, \Omega) \mid \text{div}\Psi = 0,\ \nu \times \Psi_{|\partial\Omega} = 0\}$, and introduce the orthogonal projections associated to the direct sum (4.3), $P_0 : L^2(\Omega)^6 \longrightarrow \nabla \times \widetilde{V}$ and $P_\nabla : L^2(\Omega)^6 \longrightarrow \text{grad}H^1(\Omega)$. Noting that $P_0 \nabla\phi = 0$ for all $\phi \in H^1(\Omega)$, (4.1) becomes

$$\text{(4.4)} \qquad \begin{array}{l} \textit{Find } (u,\rho) = (P_0 v + P_\nabla u, \rho) \in \mathcal{D}(\mathbb{W}) \textit{ such that} \\ \left\{ \begin{array}{l} P_0\Gamma(p)P_0 u + P_0\Gamma(p)P_\nabla u = P_0 f_1,\ x \in \Omega, \\ P_\nabla\Gamma(p)P_\nabla u + P_\nabla\Gamma(p)P_0 u - \nabla\rho = P_\nabla f_1,\ x \in \Omega, \\ n(p,x)\rho - \text{div}(P_\nabla u) = f_2\ x \in \Omega. \end{array} \right. \end{array}$$

Lemma 4.2. *The operator* $P_0\Gamma(p)P_0 \in \mathcal{B}(\nabla \times \widetilde{V})$ *is invertible with a bounded inverse for all* $p \in D_0$. *Moreover the application* $p \in D_0 \longmapsto (P_0\Gamma(p)P_0)^{-1} \in \mathcal{B}(\nabla\times\widetilde{V})$ *is holomorphic.*

Lemma 4.2 shows that $P_0 u = (P_0\Gamma(p,.)P_0)^{-1}\{-P_0\Gamma(p,.)P_\nabla u + P_0 f_1\}$, implying that one can rewrite system (4.4) into the form

$$\text{(4.5)} \qquad \begin{array}{c} \textit{Find } (w,\rho) \in \mathcal{D}(\widetilde{\mathbb{W}}) \textit{ such that} \\ \left(B(p,.) - \widetilde{\mathbb{W}}\right) \begin{pmatrix} w \\ \rho \end{pmatrix} = g, \end{array}$$

where $\widetilde{\mathbb{W}} = \begin{pmatrix} 0_{3\times 3} & \nabla \\ \text{div}P_\nabla & 0 \end{pmatrix}$ with domain $\mathcal{D}(\widetilde{\mathbb{W}}) = \left\{(u,\rho) \in \mathcal{D}(\mathbb{W})\ |u \in \text{grad}H^1(\Omega)^3\right\}$, and $w = P_\nabla u$, $g \in \text{grad}H^1(\Omega) \times L^2(\Omega)$, and $B(p,.) \in \mathcal{B}\left(L^2(\Omega)^4\right)$ is defined by

$$B(p,.) = \begin{pmatrix} P_\nabla\Gamma(p)P_\nabla - P_\nabla\Gamma(p)P_0\left(P_0\Gamma(p)P_0\right)^{-1}P_0\Gamma(p)P_\nabla & 0 \\ 0 & n(p,.) \end{pmatrix}.$$

From (A1)–(A2) and Lemma 4.2, the function $p \mapsto B(p)$ is holomorphic on D_0.

From [24], we have for all $(u,\rho) \in \mathcal{D}(\mathbb{W}) \cap (H(\text{curl},\Omega) \times L^2(\Omega))$:

$$\text{(4.6)} \qquad \|(u,\rho)\|_{H^1(\Omega)^4} \le C\left\{\|(u,\rho)\|_{L^2(\Omega)^4} + \|\mathbb{W}(u,\rho)\|_{L^2(\Omega)^4} + \|\nabla \times u\|_{L^2(\Omega)^3}\right\}.$$

Then the identity $\nabla \times \nabla\phi = 0$ and (4.6) show that the operator $(\widetilde{\mathbb{W}}, \mathcal{D}(\widetilde{\mathbb{W}}))$ has compact resolvent. Consequently, solving (4.5) is the same as inverting the holomorphic family of closed operators with compact resolvent, having a nonempty resolvent set, given by $(B(p,.) - \widetilde{\mathbb{W}}, \mathcal{D}(\widetilde{\mathbb{W}}))_{p\in D_0}$. The later is done with Fredholm analytic theory. Using (A3), there exists p_0 such that $K(p_0,.)$ is coercive. The unbounded operator $(-\mathbb{W}, \mathcal{D}(\mathbb{W}))$ is maximal dissipative so (4.2) is well-posed for $p = p_0$ and the equivalence between problems (4.2) and (4.5) shows that $B(p_0,.) - \widetilde{\mathbb{W}}$ is invertible on $\mathcal{D}(\widetilde{\mathbb{W}})$. Hence we obtain that the operator $B(p,.) - \widetilde{\mathbb{W}}$ is invertible on $\mathcal{D}(\widetilde{\mathbb{W}})$ for all $p \in D_0 \backslash S$, where S is a discrete, locally finite, and possibly empty set of D_0.

Finally, note that the inverse operator $(B(p,.) - \widetilde{\mathbb{W}})^{-1} \in \mathcal{B}(\text{grad}H^1(\Omega) \times L^2(\Omega))$ and that the application

$$p \in D_0\backslash S \longmapsto (B(p,.) - \widetilde{\mathbb{W}})^{-1} \in \mathcal{B}(\text{grad}H^1(\Omega) \times L^2(\Omega))$$

is holomorphic. From Lemma 4.2 we then derive that $(P_0 v(p,.), \rho)$ is holomorphic on $D_0\backslash S$ and hence the application $p \in D_0\backslash S \longmapsto (P_\nabla v(p,.) + P_0 v(p,.), \rho(p,.)) \in L^2(\Omega)^4$ is holomorphic too. Moreover, from the fact that $u(p,.) = a(p,.)v(p,.)$ comes the holomorphy of $p \in D_0\backslash S \longmapsto (u(p,.), \rho(p,.)) \in L^2(\Omega)^4$ and the bound

$$\forall p \in D_0\backslash S, \|(u(p,.), \rho(p,.))\|_{L^2(\Omega)^4} \le C(p)\,\|f\|_{L^2(\Omega)^4}$$

This concludes the proof. ☐

Before proving Lemma 4.2, we need the following technical result.

LEMMA 4.3. *There exists a constant $C > 0$ depending only on Ω such that*

$$\|\Phi\|_{L^2(\Omega)^3} \leq C \|\nabla \times \Phi\|_{L^2(\Omega)^3} \ \ \forall\, \Phi \in \widetilde{V}. \tag{4.7}$$

Proof of Lemma 4.3. First, one has for all $\Phi \in \widetilde{V}$ [9, 24]

$$\|\Phi\|_{(H^1(\Omega))^3} \leq C \left\{ \|\Phi\|_{(L^2(\Omega))^3} + \|\nabla \times \Phi\|_{(L^2(\Omega))^3} + \|\mathrm{div}(\Phi)\|_{L^2(\Omega)} \right\}. \tag{4.8}$$

Inequality (4.7) is then established by contradiction. Let $(\Phi_n)_n \subset \widetilde{V}$ be a sequence such that $\|\Phi_n\|_{L^2(\Omega)^3} = 1$ and $\|\nabla \times \Phi_n\|_{L^2(\Omega)^3} \leq \frac{1}{n}$. From (4.8) $(\Phi_n)_n$ is bounded in the H^1 norm, hence it has a strongly convergent subsequence in the $L^2(\Omega)^3$ norm toward some $\Phi_0 \in H^1(\Omega)^3$. Since $\Phi_n \in \widetilde{V}$, for all n, one obtains $\mathrm{div}\Phi_0 = 0$ in the sense of distributions and $\nu \times \Phi_{0|\partial\Omega} = 0$ in $H^{-\frac{1}{2}}(\partial\Omega)^3$. Moreover, as $\|\nabla \times \Phi_n\|_{L^2(\Omega)^3} \leq \frac{1}{n}$ it comes $\nabla \times \Phi_0 = 0$. Consider now the following Hodge decomposition [9, p. 353]:

$$L^2(\Omega)^3 = H_0(\mathrm{curl}0, \Omega) \oplus H(\mathrm{div}0, \Omega),$$

where $H_0(\mathrm{curl}0, \Omega) = \{v \in L^2(\Omega)^3 \mid \nabla \times v = 0 \text{ on } \Omega,\ \nu \times v = 0 \text{ on } \partial\Omega\}$. We thus deduce that Φ_0 belongs to both $H_0(\mathrm{curl}0, \Omega)$ and $H(\mathrm{div}0, \Omega)$ so $\Phi_0 = 0$, which contradicts the fact that $\|\Phi_0\|_{L^2(\Omega)^3} = 1$. ◻

Remark 4.4. Lemma 4.3 is recovered from inequality (4.8) by compactness but holds without (4.8) (see [13, p. 553]).

Proof of Lemma 4.2. First note that inverting $P_0\Gamma(p)P_0$ on $\nabla \times \widetilde{V}$ is the same as solving the following boundary value problem:

$$\begin{aligned} &\text{Find } \Phi \in \widetilde{V} \text{ such that for all } \Psi \in \widetilde{V} \\ &\int_\Omega \left\langle \Gamma(p)\nabla \times \Phi, \overline{\nabla \times \Psi} \right\rangle dx = \langle h, \Psi \rangle_{\widetilde{V}' \times \widetilde{V}}, \end{aligned} \tag{4.9}$$

where h belongs to $\widetilde{V}'$, which is the set of continuous linear forms on $\widetilde{V}$. Integrating by parts and using the boundary condition, Φ is thus solution to the equation

$$\begin{aligned} &\text{Find } \Phi \in \widetilde{V} \text{ such that} \\ &\begin{cases} \nabla \times \nabla \times \Phi = h(p, .) \text{ in } \Omega \\ \nu \times \Phi_{|\partial\Omega} = 0 \text{ on } \partial\Omega, \end{cases} \end{aligned}$$

where $h(p) = h/\Gamma(p) \in \widetilde{V}'$. Note from inequality (4.8) that the set $\widetilde{V}$ is a Hilbert space when equipped with the usual $H^1(\Omega)^3$ norm. The coercivity of the bilinear form involved in (4.9) thus follows from Lemma 4.3, and the proof of the first part of Lemma 4.2 is then done by using the Lax–Milgram lemma. The holomorphy of $p \in D_0 \longmapsto (P_0\Gamma(p)P_0)^{-1} \in \mathcal{B}(\nabla \times \widetilde{V})$ then comes from [19] and the holomorphy of $\Gamma(p)$. ◻

The range of application of the result presented into Theorem 4.1 seems to be quite restricted since it requires the physical parameters to not depend on x. However, it can be extended with the following assumption.

Assumption 9 (for nonconstant and tensorial acoustic materials). Suppose that (A1), (A2), (A3), and (A4) hold with the additional following condition:

(A4) There exists $a(p, x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, satisfying $a(p, .)_{|\partial\Omega} = 1$ and such that the following inequality holds:

$$\mathcal{R}e \left\langle \Gamma(p, x)a(p, x)X, \overline{X} \right\rangle \geq \alpha |X|^2$$

for all $p \in D_0$, for almost all $x \in \Omega$, and for all $X \in \mathbb{C}^3$.

COROLLARY 4.5. *If Assumption* 9 *is satisfied, then* (4.1) *is well-posed for all* $p \in D_0 \backslash S$*, where* $S \subset D_0$ *is a discrete, locally finite, and possibly empty set of* D_0*. Moreover, the solution satisfies the bound*

$$\forall p \in D_0 \backslash S, \|(u(p,.), \rho(p,.))\|_{L^2(\Omega)^4} \leq C(p) \|f\|_{L^2(\Omega)^4},$$

and the application $p \in D_0 \backslash S \longmapsto (u(p,.), \rho(p,.)) \in L^2(\Omega)^4$ *is holomorphic.*

Proof. We follow the proof of Theorem 4.1. However, before projecting (4.2) according to the Hodge decomposition (4.3), we perform the change of unknown $u = a(p,.)v$. Then one derives from (A4) that (v, ρ) belongs to $\mathcal{D}(\mathbb{W})$ and is solution to

$$\text{Find } (v,\rho) \in \mathcal{D}(\mathbb{W}) \text{ such that} \\ \left\{ \begin{array}{l} \Gamma(p,.)a(p,x)v(p,x) - \nabla\rho(p,x) = f_1(x), \ x \in \Omega, \\ a^{-1}(p,x)n(p,x)\rho(p,x) + a(p,x)^{-1}\langle \nabla a(p,x), v(p,x)\rangle \\ \qquad\qquad - \operatorname{div}(v(p,x)) = a^{-1}(p,x) f_2(x) \text{ for } x \in \Omega. \end{array} \right. \tag{4.10}$$

Then projecting (4.10),

$$\text{Find } (v,\rho) = (P_0 v + P_\nabla v, \rho) \in \mathcal{D}(\mathbb{W}) \text{ such that} \\ \left\{ \begin{array}{l} P_0\Gamma(p,x)a(p,x)P_0 v + P_0\Gamma(p,x)a(p,x)P_\nabla v = P_0 f_1, \ x \in \Omega, \\ P_\nabla\Gamma(p,x)a(p,x)P_\nabla v + P_\nabla\Gamma(p,x)a(p,x)P_0 v - \nabla\rho = P_\nabla f_1, \ x \in \Omega, \\ a^{-1}(p,x)n(p,x)\rho(p,x) + a(p,x)^{-1}\langle \nabla a(p,x), v(p,x)\rangle \\ \qquad\qquad - \operatorname{div}(P_\nabla v) = a^{-1}(p,x) f_2(x), \ x \in \Omega. \end{array} \right. \tag{4.11}$$

LEMMA 4.6. *The operator* $P_0\Gamma(p,.)a(p,.)P_0 \in \mathcal{B}(\nabla \times \widetilde{V})$ *is invertible with a bounded inverse for all* $p \in D_0$ *and the application* $p \in D_0 \longmapsto (P_0\Gamma(p,.)a(p,.)P_0)^{-1} \in \mathcal{B}(\nabla \times \widetilde{V})$ *is holomorphic.*

Lemma 4.6 shows that $P_0 v = (P_0\Gamma(p,.)a(p,.)P_0)^{-1}\{-P_0\Gamma(p,.)a(p,.)P_\nabla v + P_0 f_1\}$, implying system (4.11) to be rewritten into the form

$$\text{Find } (P_\nabla v, \rho) \in \mathcal{D}(\widetilde{\mathbb{W}}) \text{ such that} \\ \left(B(p,.) - \widetilde{\mathbb{W}}\right)\begin{pmatrix} P_\nabla v \\ \rho \end{pmatrix} = g \text{ on } \Omega,$$

where $g \in \operatorname{grad} H^1(\Omega) \times L^2(\Omega)$, $B(p,.) \in \mathcal{B}(L^2(\Omega)^4)$ is a bounded operator which is holomorphic on D_0 (thanks to (A1), (A2), (A4), and Lemma 4.6). The end of the proof is now the same as that for Theorem 4.1. ∎

Proof of Lemma 4.6. Following the proof of Lemma 4.2, we have that inverting $P_0\Gamma(p,.)a(p,.)P_0$ on $\nabla \times \widetilde{V}$ remains to find a $\Phi \in \widetilde{V}$ solution to

$$\text{Find } \Phi \in \widetilde{V} \text{ such that} \\ \left\{ \begin{array}{l} \nabla \times \Gamma(p,x)a(p,x)\nabla \times \Phi(p,x) = h, \ x \in \Omega, \\ \nu \times \Phi(p,x) = 0 \text{ on } \partial\Omega. \end{array} \right.$$

From assumption (A4), it follows for all $p \in D_0$ that the multiplicative operator $a(p,.)\Gamma(p,.)$ is coercive. The bilinear form $(\Phi, \Psi) \longmapsto \int_\Omega \left\langle \Gamma(p,.)a(p,.)\nabla \times \Phi, \overline{\nabla \times \Psi}\right\rangle dx$ is then coercive on $\widetilde{V}$ equipped with the usual $H^1(\Omega)^3$ norm (see Lemma 4.3). Consequently the Lax–Milgram theorem shows the first part of the lemma. The holomorphy of $p \in D_0 \longmapsto (P_0\Gamma(p,.)a(p,.)P_0)^{-1} \in \mathcal{B}(\nabla \times \widetilde{V})$ comes from the holomorphy of $\Gamma(p,.)a(p,.)$ (thanks to (A1), (A2), (A4)) [19]. ∎

5. Generic well-posedness for linear elasticity. The physical properties of the elastic metamaterial manifest with, for instance, negative mass density or bulk modulus [10, 36]. Hence, we consider the equations of elastodynamics [6]:

$$\tag{5.1}\begin{aligned}&\textit{Find } u=(u_1,u_2,u_3)\in H^1(\Omega)^3 \textit{ such that for } j=1,2,3\\ &\begin{cases}\operatorname{div}(\mu(p,x)\nabla u_j(p,x))+\partial_j(\lambda(p,x)+\mu(p,x))\operatorname{div}(u(p,x))\\ \qquad\qquad -p^2u_j(p,x)=f(x) \text{ on } \Omega,\\ \left\langle(\mu(p,x)\nabla u_j)_{|\partial\Omega},\nu\right\rangle+\nu_j(\lambda(p,x)+\mu(p,x))_{|\partial\Omega}\operatorname{div}(u)_{|\partial\Omega}\\ \qquad\qquad -(\Lambda(x)u_j)_{|\partial\Omega}=0 \text{ on } \partial\Omega,\end{cases}\end{aligned}$$

where λ,μ are the Lamé coefficients, f is the body force per unit volume, $\Lambda:\partial\Omega\longrightarrow\mathbb{C}$ is an impedance assumed to be coercive, and u is the displacement. To formulate (5.1) as a first order system of a partial differential equation, we introduce new unknowns:

$$\begin{cases} v_j=\mu(p,x)\nabla u_j,\ \ j=1,2,3,\\ \gamma=(\lambda(p,x)+\mu(p,x))\operatorname{div}u.\end{cases}$$

Thus for $j=1,2,3$, (5.1) reduces to the following first order system:

$$\tag{5.2}\begin{aligned}&\textit{Find } (v_j,u_j,\gamma)\in H(\operatorname{div},\Omega)\times H^1(\Omega)\times H^1(\Omega) \textit{ such that}\\ &\begin{cases}p^2u_j-\operatorname{div}(v_j)-\partial_j\gamma=-f \text{ on } \Omega,\\ \mu^{-1}(p,x)v_j-\nabla u_j=0 \text{ on } \Omega,\\ (\lambda(p,x)+\mu(p,x))^{-1}\gamma-\operatorname{div}(u)=0 \text{ on } \Omega,\\ \langle v_j(x),\nu\rangle+\nu_j\gamma(x)-\Lambda(x)u_j(x)=0 \text{ on } \partial\Omega.\end{cases}\end{aligned}$$

The conditions to be verified by the metamaterial are thus as follows.

Assumption 10 (for elastic materials).

- (E1) The applications $\lambda(p,x)^{-1}$ and $(\lambda(p,x)+\mu(p,x))^{-1}$ are holomorphic on D_0 for almost all $x\in\Omega$, where D_0 is a connected open set of $\mathbb{C}$.
- (E2) The applications $\lambda(p,x)^{-1}$ and $(\lambda(p,x)+\mu(p,x))^{-1}$ belong to $L^\infty(\overline{\Omega})$ for all $p\in D_0$.
- (E3) There exists p_0 in D_0 and $\alpha>0$ such that the following inequality holds:

$$\begin{aligned}&\mathcal{R}e\left\{\langle p_0^2X,\overline{X}\rangle+\mathcal{R}e\langle\mu^{-1}(p_0,x)Y,\overline{Y}\rangle\right\}\\ &\quad+\mathcal{R}e(\lambda(p_0,x)+\mu(p_0,x))^{-1}|z|^2\geq\alpha(|X|^2+|Y|^2+|z|^2)\end{aligned}$$

for all $(X,Y,z)\in\mathbb{C}^3\times\mathbb{C}^3\times\mathbb{C}$ and for almost all $x\in\Omega$.

We then have the following result.

THEOREM 5.1. *Suppose that $\mu(p,.)\neq 0$ (see (5.2)) is scalar valued and does not depend on x. If Assumption 10 is fulfilled, then system (5.2) is well-posed for all $p\in D_0\backslash S$, where $S\subset D_0$ is a discrete, locally finite, and possibly empty set of D_0. Moreover, the solution is continuous with respect to the data, and the application $p\in D_0\backslash S\longmapsto(v(p,.),u(p,.),\gamma(p,.))\in L^2(\Omega)^{13}$ is holomorphic.*

Proof. First, let us introduce the following unbounded operator:

$$\mathbb{E}=\begin{pmatrix}0&0&0&\nabla&0&0&0\\0&0&0&0&\nabla&0&0\\0&0&0&0&0&\nabla&0\\ \operatorname{div}&0&0&0&0&0&\partial_1\\0&\operatorname{div}&0&0&0&0&\partial_2\\0&0&\operatorname{div}&0&0&0&\partial_3\\0&0&0&\partial_1&\partial_2&\partial_3&0\end{pmatrix},$$

acting on vector fields of the form $(v_1, v_2, v_3, u_1, u_2, u_3, \gamma) \in \mathbb{C}^{13}$ on the domain $\mathcal{D}(\mathbb{E})$

$$\mathcal{D}(\mathbb{E}) = \left\{ \forall j,\, (v_j, u_j, \gamma) \in H(\mathrm{div}, \Omega) \times H^1(\Omega) \times H^1(\Omega) \middle| \, (v_j, u_j, \gamma)|_{\partial\Omega} \in \ker \widetilde{N}_j(x) \right\},$$

where $\widetilde{N}_j(x)(v_j, u_j, \gamma) = \langle (v_j(x), \nu \rangle + \nu_j \gamma(x) - \Lambda u_j(x)$. Thus (5.2) collapses as

$$\text{(5.3)} \qquad \begin{array}{l} \textit{Find } \Pi = (v_1, v_2, v_3, u_1, u_2, u_3, \gamma) \in \mathcal{H}_{\mathbb{E}} \textit{ such that} \\ \begin{cases} (K(p,x) - \mathbb{E})\, \Pi = F, \; x \in \Omega, \\ \langle (v_j)_{|\partial\Omega}, \nu \rangle + \nu_j \gamma_{|\partial\Omega} - (\Lambda(x) u_j)_{|\partial\Omega} = 0 \text{ on } \partial\Omega, \end{cases} \end{array}$$

where $F \in L^2(\Omega)^{13}$ and the multiplicative operator is given by

$$\text{(5.4)} \qquad K(p,x) = \begin{pmatrix} \mu^{-1}(p)\mathbb{I}_9 & 0\mathbb{I}_3 & 0 \\ 0\mathbb{I}_9 & p^2\mathbb{I}_3 & 0 \\ 0\mathbb{I}_9 & 0\mathbb{I}_3 & (\lambda(p,x) + \mu(p,.))^{-1} \end{pmatrix}.$$

The operator $(-\mathbb{E}, \mathcal{D}(\mathbb{E}))$ being maximal dissipative, (5.3) is well-posed for $p = p_0$ [31].

At last, (5.2) is similar to the first order wave equation (4.1). Hence, one simply needs to control the curl of vector fields $v_j \in H(\mathrm{div}, \Omega)$ to recover some compactness for the resolvent of $(\mathbb{E}, \mathcal{D}(\mathbb{E}))$. Thus, using the results of [24], the following inequality holds for all $\Pi = (v_1, v_2, v_3, u_1, u_2, u_3, \gamma) \in \mathcal{D}(\mathbb{E}) \cap (H(\mathrm{curl}, \Omega))^3 \times L^2(\Omega)^4$:

$$\|\Pi\|_{H^1(\Omega)^{13}} \le C \left(\|\Pi\|_{L^2(\Omega)^{13}} + \|\mathbb{E}\Pi\|_{L^2(\Omega)^{13}} + \sum_{j=1}^{3} \|\nabla \times v_j\|_{L^2(\Omega)^3} \right).$$

From now, we mimic the proof of Theorem 4.1. First, projecting (5.3) with the help of Hodge decomposition (4.3), and using the same notation as for the acoustics, we infer that

$$P_0 v_j = \left(P_0 \mu(p,.)^{-1} P_0 \right)^{-1} \left\{ -P_0 \mu(p,.)^{-1} P_\nabla v_j + P_0 G_j \right\}$$

for $G_j \in L^2(\Omega)^3$ and $j = 1, 2, 3$. This implies that system (4.2) is equivalent to

$$\text{(5.5)} \qquad \begin{array}{l} \textit{Find } \widetilde{\Pi} = (P_\nabla v_1, P_\nabla v_2, P_\nabla v_3, u_1, u_2, u_3, \gamma) \in \mathcal{D}(\widetilde{\mathbb{E}}) \textit{ such that} \\ \left(B(p,.) - \widetilde{\mathbb{E}} \right) \Pi = \widetilde{F}, \end{array}$$

where $\widetilde{F} \in (\mathrm{grad} H^1(\Omega))^3 \times L^2(\Omega)^4$, $B(p,.) \in \mathcal{B}(L^2(\Omega)^{13})$ is a bounded operator which is also holomorphic on D_0, and $\widetilde{\mathbb{E}}$ is defined by

$$\widetilde{\mathbb{E}} = \begin{pmatrix} 0 & 0 & 0 & \nabla & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \nabla & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nabla & 0 \\ \mathrm{div} P_\nabla & 0 & 0 & 0 & 0 & 0 & \partial_1 \\ 0 & \mathrm{div} P_\nabla & 0 & 0 & 0 & 0 & \partial_2 \\ 0 & 0 & \mathrm{div} P_\nabla & 0 & 0 & 0 & \partial_3 \\ 0 & 0 & 0 & \partial_1 & \partial_2 & \partial_3 & 0 \end{pmatrix}$$

with domain $\mathcal{D}(\widetilde{\mathbb{E}}) = \mathcal{D}(\mathbb{E}) \cap (\mathrm{grad} H^1(\Omega))^3 \times L^2(\Omega)^4$. From (4.8), the embedding of $\mathcal{D}(\widetilde{\mathbb{E}})$ into $L^2(\Omega)^{13}$ is compact. It yields the compactness of the resolvent of $(\widetilde{\mathbb{E}}, \mathcal{D}(\widetilde{\mathbb{E}}))$. Since this operator is maximal dissipative and $B(p,.)$ is a bounded multiplicative operator of $L^2(\Omega)^{13}$ for all $p \in D_0$, solving (5.5) is the same as inverting a holomorphic family of closed operators with compact resolvent, which can be done with Fredholm analytical theory. The end of the proof is identical to the one of Theorem 4.1. $\square$

As for Theorem 4.1 we can here extend the previous results with the following.

Assumption 11 (for nonconstant and tensorial elastic materials). Suppose that (E1), (E2), (E3), and (E4) hold with the following additional condition.

(E4) There exists $\alpha > 0$ and $a(p,x) \in \mathbb{C}$, Lipschitz continuous for $x \in \overline{\Omega}$ and holomorphic for $p \in D_0$, satisfying $a(p,.)_{|\partial\Omega} = 1$, such that

$$\mathcal{R}e\left(\langle \mu(p,x)^{-1} a(p,x) X, \overline{X} \rangle\right) \geq \alpha |X|^2$$

for all $p \in D_0$, for almost all $x \in \Omega$, and for all $X \in \mathbb{C}^3$.

Corollary 5.2. *If Assumption* 11 *is verified, then* (5.2) *is well-posed for all* $p \in D_0 \backslash S$, *where* $S \subset D_0$ *is a discrete, locally finite, and possibly empty set of* D_0. *Moreover, the solution is continuous with respect to the data, and the application* $p \in D_0 \backslash S \longmapsto (v(p,.), u(p,.), \gamma(p)) \in L^2(\Omega)^{13}$ *is holomorphic.*

Proof. We follow the proof of Corollary 4.5. Perform the changes of unknown $X_j(p) = a(p,.)v_j$ and $\gamma(p) = a(p,.)\widetilde{\gamma}$ and then project the system according to the Hodge decomposition (4.3). Thus, from (E4),

$$P_0 X_j = \left(P_0 \mu(p,.)^{-1} a(p,.) P_0\right)^{-1} \left\{ -P_0 \mu(p,.)^{-1} a(p,.) P_\nabla X_j + P_0 G_j \right\}$$

for $j = 1, 2, 3$. This implies that system (4.2) is equivalent to

$$\begin{array}{l} \textit{Find } \widetilde{\Pi} = (P_\nabla X_1, P_\nabla X_2, P_\nabla X_3, u_1, u_2, u_3, \widetilde{\gamma}) \in \mathcal{D}(\widetilde{\mathbb{E}}) \textit{ such that} \\ \left(B(p,.) - \widetilde{\mathbb{E}}\right) \Pi = \widetilde{F}, \end{array}$$

where $F \in (\mathrm{grad} H^1(\Omega))^3 \times L^2(\Omega)^4$, $B(p,.) \in \mathcal{B}(L^2(\Omega)^{13})$ is a bounded operator which is also holomorphic on D_0. The end of the proof is the same as that for Theorem 5.1. $\square$

6. Study of some examples.

This section illustrates our results with some examples from the literature. We successively apply them to the study of Maxwell's equations with a periodical array of SRR, a Drude–Born–Fedorov system with some chiral metamaterial made from the Ω-particle resonator model or with a bi-anisotropic metamaterials. Examples that are also considered are the wave equation with some absorbing boundary condition of perfectly matched layers (PML) type and the wave equation with a homogenized acoustic metamaterial having negative bulk modulus.

6.1. Periodic array of SRR.

The SRR were introduced by Pendry in 2000 as the first example of negative index material [30, 33]. Some studies dealing with a periodic array of SRR have followed [18, 29, 32], but the well-posedness of this system remains unanswered to the best of our knowledge. The effective parameters involved in (3.1) of a periodical array of interspaced conducting nonmagnetic SRR and continuous wires calculated in [32] have the following expressions:

$$\left\{ \begin{array}{l} \varepsilon(p,x) = \left(1 + \frac{w_G^2}{p^2}\right) \mathbb{I}_3, \\ \mu(p,x) = \left(1 + \frac{Fp^2}{-p^2 - w_0^2 + p\Gamma}\right) \mathbb{I}_3, \\ \beta(p,x) = 0, \end{array} \right. \tag{6.1}$$

where $w_G > 0$ is the plasma pulsation of gold and $w_0 = \sqrt{\frac{3l}{\pi^2 \mu_0 C r^3}}$ and $\Gamma = \frac{2l\rho}{r\mu_0} > 0$ are constants. Here ρ is the resistance per unit length of the rings measured around the circumference, l is the distance between layers, a is the lattice parameter, r is a geometrical parameter (defined in [32, Figure 1]), and C is the capacitance associated with the gaps between the rings.

We consider here a homogenized SRR embedded in a connected bounded open set $\Omega \subset \mathbb{R}^3$ with connected $\mathcal{C}^1$ boundary. We invoke Theorem 3.2 to study (3.1)–(6.1). So we have to check Assumption 3.

(B1) ε and μ defined by (6.1) behave on p as rational fractions and thus are holomorphic on $D_0 = \mathbb{C}\backslash Z$, where Z is the set of zeros of their denominators. Namely, $Z = \{(\Gamma - \sqrt{\Gamma^2 - 4w_0^2})/2, 0, (\Gamma + \sqrt{\Gamma^2 - 4w_0^2})/2\}$, where $\sqrt{y}$ is equal to $i\sqrt{-y}$ when $y < 0$.

(B2) Coefficients ε and μ are constant in x and thus have the requested regularity.

(B3) Let $g : p \in \mathbb{R}\backslash\{0\} \longmapsto -p^2 - w_0^2 + p\Gamma \in \mathbb{R}$. The maximum of g is reached for $p_0 = \frac{\Gamma}{2}$. As $g(p_0) > 0$, (B3) is satisfied.

Hence, Theorem 3.2 implies the well-posedness of Maxwell's equations in the presence of homogenized SRR (3.1)–(6.1) with $f \in H(\mathrm{div}, \Omega)^2$ for all p in $D_0\backslash\widetilde{S}$, where $\widetilde{S}$ is a discrete, locally finite, and possibly empty set of D_0.

6.2. A bi-anisotropic metamaterial. We consider now a material described by a lattice composed of a bi-anisotropic homogenized SRR as studied in [21]. The physical parameters of the homogenized material are now defined by (3.5) with the following parameters:

$$\begin{cases} \varepsilon(p) = 1 + (\frac{dc_0}{l^2})^2 \frac{F}{w_{LC}^2 + p^2 - p\gamma}, \\ \mu(p,.) = 1 - \frac{Fp^2}{w_{LC}^2 + p^2 - p\gamma}, \\ \xi(p) = i\frac{dc_0}{l^2}\frac{Fp}{w_{LC}^2 + p^2 - p\gamma}, \\ \zeta(p,.) = -i\frac{dc_0}{l^2}\frac{Fp}{w_{LC}^2 + p^2 - p\gamma}, \end{cases} \tag{6.2}$$

where l and d are positive constant, c_0 is the speed of light in the vacuum, F is the SRR volume filling fraction, $\gamma > 0$ is the damping, and $w_{LC} > 0$ is the LC eigenfrequency.

Theorem 3.2 will be used to study the system (3.2)–(6.2), and hence Assumption 3 has to be checked.

(B1) Coefficients (6.2) depend on p like rational fractions. So, they are holomorphic on $\mathbb{C}\backslash Z$ with $Z = \{(\gamma \pm \sqrt{\gamma^2 - w_{LC}^2})/2\}$.

(B2) $\varepsilon(p,x), \mu(p,x), \xi(p,x)$, and $\zeta(p,x)$ are obviously Lipschitz continuous in $x \in \overline{\Omega}$ for all $p \in \mathbb{C}\backslash Z$. To show that $\varepsilon(p,x)\mu(p,x) - \xi(p,x)\zeta(p,x)$ does not vanish in a well-suited domain $D_0 := \mathbb{C}\backslash(Z \cup \{p_i, i = 1\cdots 5\})$, we compute the zeros p_i (which do not depend on x) in p of this function using the explicit values of $dc_0/l^2 = 0.75w_{LC},\ F = 0.3$, and $\gamma = 0.05w_{LC}$ (see [21, Figure 2]):

$$\begin{cases} p_1 = (0.1522392099 + 1.128302032i)w_{LC}, \\ p_2 = (-0.09152492420 + 1.131232555i)w_{LC}, \\ p_3 = (-0.09152492420 - 131232555i)w_{LC}, \\ p_4 = (0.1522392099 + 1.128302032i)w_{LC}, \\ p_5 = 0. \end{cases}$$

(B3) To fulfill (B3) one needs to find a $p_0 \in D_0$ satisfying the constraint given in (B3). It can be done by looking for a p_0 such that $K(p)$ (3.5) is coercive at this point. Its spectrum is

$$\sigma(K(p,.)) = \Big\{ \Big(\varepsilon(p) + \mu(p,.) - \sqrt{(\varepsilon(p)^2 - 2\mu(p,.)\varepsilon(p) + \mu(p,.)^2 + 4\xi(p)^2)} \Big) \frac{p}{2}, \Big(\varepsilon(p) + \mu(p,.) + \sqrt{(\varepsilon(p)^2 - 2\mu(p,.)\varepsilon(p) + \mu(p,.)^2 + 4\xi(p)^2)} \Big) \frac{p}{2} \Big\}.$$

For $p = w_{LC}$ we have $K(w_{LC}) = K(w_{LC})^*$ and

$$\sigma(K(w_{LC})) = \{0.7997334285 w_{LC},\ 1.132958880 w_{LC}\},$$

both with multiplicity 3. Finally $K(w_{LC})$ is coercive.

Consequently, Theorem 3.2 can be applied to show that Maxwell's equation (3.2) in the presence of a bi-anisotropic material described by a homogenized SRR (6.2) is well-posed with right member $f \in (H(\text{div}, \Omega))^2$ for all p in $D_0 \backslash S$, where S in an exceptional set of values.

6.3. Chiral metamaterial based on the Ω-particle resonator model. We consider a material described by a Ω-particle resonator model (see [35, Figure 6]) embedded in a connected bounded open set $\widetilde{\Omega}$ (the notation changes here to avoid confusion) which is filled with a material whose positive parameters are denoted $\varepsilon_b,\ \mu_b$. The electromagnetic properties are described by (3.3) with the following parameters:

$$\left\{ \begin{array}{l} \varepsilon(p,x) = \varepsilon_b + \frac{\Omega_\varepsilon w_0^2}{w_0^2 + p^2 - p\gamma} \mathbb{I}_3, \\ \mu(p,x) = \mu_b - \frac{\Omega_\mu p^2}{w_0^2 + p^2 - p\gamma} \mathbb{I}_3, \\ \beta(p,x) = \frac{\Omega_\beta p}{i(w_0^2 + p^2 - p\gamma)} \mathbb{I}_3, \end{array} \right. \tag{6.3}$$

where $w_0,\ \gamma$, and $\Omega_{\varepsilon,\mu,\beta}$ are some positive constants defined by the homogenized model [35].

To be used on system (3.3)–(6.3), Theorem 3.2 now requires checking Assumption 4:

(C1) The permittivity, the permeability, and the chirality defined by (6.3) are rational fractions of p. Hence (C1) is satisfied on $\mathbb{C} \backslash Z$, where $Z = \{(\gamma \pm \sqrt{\gamma^2 - w_0^2})/2\}$ is the set of zeros of their denominators.

(C2) ε, μ, and β do not depend on x, hence the required regularity is obviously satisfied. The quantity $p^2 \varepsilon(p,x)\mu(p,x)M(p,x)$ does not vanish since $p \notin Z \cup \{p_j, j = 1 \cdots 5\}$, where $(p_j)_j$ are the zeros of this function. Using the explicit values of $\Omega_{\varepsilon,\mu,\beta}$ and ε_b, μ_b (see [35, p. 12]) we find

$$\left\{ \begin{array}{l} p_1 = 0,05146743450 - 1.909033734i, \\ p_2 = 0,05146743450 + 1.909033734i, \\ p_3 = 0,05498575999 - 1.927011242i, \\ p_4 = 0,05498575999 + 1.927011242i, \\ p_5 = 0. \end{array} \right.$$

(C3) We use again the notation $K(p)$ (3.5) and infer its spectrum to be $\sigma(K(p))$

$$= \left\{ \frac{\varepsilon(p) + \mu(p) - \sqrt{\varepsilon(p)^2 - 2\varepsilon(p)\mu(p) + \mu(p)^2 - 4\varepsilon(p)^2\mu(p)^2p^2\beta(p)^2}p}{2 + 2p^2\varepsilon(p)\mu(p)\beta(p)} \right.$$

$$\left. \frac{\varepsilon(p) + \mu(p) + \sqrt{\varepsilon(p)^2 - 2\varepsilon(p)\mu(p) + \mu(p)^2 - 4\varepsilon(p)^2\mu(p)^2)p^2\beta(p)^2}p}{2 + 2p^2\varepsilon(p)\mu(p)\beta(p)} \right\},$$

both with multiplicity 3. Thus, the spectrum of $K(1) = K(1)^*$ is $\sigma(K(1)) = \{0.9536351793, 3.444237487\}$, both with multiplicity 3, and then $K(p_0)$ is coercive for $p_0 = 1$.

Consequently, Theorem 3.2 can be applied to show that Maxwell's equation (3.3) in the presence of a chiral material described by a Ω-particle resonator model (6.3) is well-posed with right member $f \in (H(\mathrm{div}, \widetilde{\Omega}))^2$ for all p in $D_0 \backslash S$, where S is an exceptional set of values.

6.4. Acoustic metamaterial with negative modulus. We consider now a homogenized acoustic metamaterial made from short tubes with a side hole used as unit cell like the one studied in [23]. The homogenized media has negative modulus at some frequencies. It has been remarked in [23] that such material behaves analogously to the one having negative permittivity. The physical modeling is given by system (4.1) with parameters

$$\Gamma(p,x) = p\vartheta\mathbb{I}_3, \quad n(p,x) = pB^{-1}\left(1 - \frac{w_{sh}^2}{\gamma p - p^2}\right), \tag{6.4}$$

where $x \in \Omega$, γ is the damping term, $\vartheta = 1.21$ kg/m^3, $B = 1.42 \times 10^5$ Pa is the bulk modulus of air, and $w_{sh} > 0$ is defined in [23].

According to Theorem 4.1 we are going to check Assumption 8.

(A1) Coefficients in (6.4) are meromorphic with pole at 0 and γ and hence they are holomorphic in $D_0 = \mathbb{C}\backslash \{0, \gamma\}$.

(A2) Γ and n are not depending on x; hence they have the required regularity and invertiblity since they are defined for any x when p belongs to D_0.

(A3) For $p_0 \in \mathbb{R}^+$ large enough, $p_0B^{-1}(1 - \frac{w_{sh}^2}{\gamma p_0 - p_0^2}) > 0$, thus satisfying the condition.

Finally, Theorem 4.1 can be applied to show that (4.1)–(6.4) is well-posed for all $p \in D_0 \backslash S$, where S is a discrete, locally finite and possibly empty set of D_0.

6.5. Approximate cloaking for the wave equation. It is well known that perfect acoustic or electromagnetic cloaking is hard since it uses singular transformations optics, which lead to singular material parameters (see [14] and references therein). This difficulty has been overcome recently with the *approximate* cloaking (see [15] and references therein) giving rise to nonsingular parameters which are bounded from above and below. However, most approximate cloaking devices do not use materials with frequency-dependent parameters such as metamaterials. Nevertheless, there exist theoretical materials defined as complex absorbing boundary conditions [27] which perform approximate cloaking since they absorb strongly the scattered waves. We thus consider them as metamaterial in a broader sense. Actually, as an application of our theory, we are going to retrieve that the wave equation (4.1) in the presence of a complex absorbing boundary condition of PML type is well-posed.

Now we introduce the construction of PML via a complex stretching of $\mathbb{R}^3$ [27]. Let Θ be a convex subset of Ω with $\mathcal{C}^3$ boundary whose outward unitary normal is noted $\mathbf{n}_\Theta$. Since Θ is convex, there exists for all $x \in \Omega$ a unique $\eta_x \in \partial\Theta$ such that $h = dist(x, \partial\Theta) = |x - \eta_x|$ and $x = \eta_x + h\mathbf{n}_\Theta(\eta_x)$. Let $\sigma \in C^3(\mathbb{R}^+, \mathbb{R}^+)$ such that $\lim_{s \longrightarrow +\infty} \sigma(s) = \lim_{s \longrightarrow +\infty} \sigma'(s) = +\infty$ and $\sigma(0) = \sigma'(0_+) = 0$. A complex formulation of the PML is then obtained introducing the new coordinates:

$$\widetilde{x} : \mathbb{R}^3 \longrightarrow \mathbb{C}^3, \; x \longmapsto \widetilde{x}(x) = x + \frac{\sigma(h)\mathbf{n}_\Theta(\eta_x)}{p}. \tag{6.5}$$

At last, noting $J(p, x) = \nabla\widetilde{x}(p, x)$ the Jacobian of this transformation, we assume that (6.5) defines a bijective mapping for $(p, x) \in \mathbb{C}\backslash\{0\} \times \Omega$. The wave equation with absorbing boundary condition of PML type is then (4.1) with coefficients

$$\Gamma(p, x) = \sqrt{det(g(p, x))}\, g(p, x)^{-1}, \quad n(p, x) = p^2 det(g(p, x)), \tag{6.6}$$

where $g(p, x) = (J(p, x)^T J(p, x))^{-1}$.

We show that (4.1)–(6.6) is well-posed with Corollary 4.5 by checking Assumption 9.

- (A1) For almost all $x \in \Omega$, the application $p \in D_0 = \mathbb{C}\backslash\{0\} \longmapsto g(p, x) \in \mathrm{Hom}(\mathbb{C}^3)$ is holomorphic, and so are $p \mapsto \Gamma(p, .)$ and $p \mapsto n(p, .)$.
- (A2) For any $p \in D_0 := \mathbb{C}\backslash\{0\}$, $g(p, .)$ is invertible. Moreover, $g(p, .)$ and $g(p, .)^{-1}$ are bounded on Ω since the application σ belongs to $\mathcal{C}^3(\mathbb{R}^+, \mathbb{R}^+)$ and never vanishes and since $x \mapsto \widetilde{x}(x)$ (6.5) is bijective by hypothesis.
- (A3) The tensor $g(p_0)$ is coercive with $p_0 = 1$. Hence, $(X, z) \mapsto |z|^2 \mathcal{R}e(n(1, x)) + \mathcal{R}e\{\langle \Gamma(1, x)X, \overline{X}\rangle\}$ is coercive too.
- (A4) Pose $a(p, x) = 1/\sqrt{det(g(p, x))}$; then $\langle \Gamma(p, x)a(p, x)X, \overline{X}\rangle = |J(p, x)X|^2$, which is coercive.

Hence, the system (4.1)–(6.6) is well-posed for all $p \in D_0\backslash S$, where S is a discrete, locally finite, and possibly empty set of $D_0 = \mathbb{C}\backslash\{0\}$.

Remark 6.1. We remark, in the same way, that we can prove that Maxwell's equations with absorbing boundary condition of PML type are well-posed too. Indeed the presence of convex PML implies that the physical parameters (see [27] for computations) in (3.1) are modified as follows:

$$\varepsilon(p, x) = \mu(p, x) = g(p, x)/\sqrt{\det(g(p, x))}. \tag{6.7}$$

The same study can now be led using Theorem 3.6. Thus, we have to check Assumption 5, where (BT1), (BT2), (BT3) are already proved with (A1), (A2), (A3). The last item, (BT4), is satisfied taking $a_\varepsilon(p, x) = a_\mu(p, x) = \sqrt{\det(g(p, x))}$. Hence, Maxwell's equations (3.1)–(6.7) with PML are well-posed for all $p \in D_0\backslash S$, where S is a discrete, locally finite, and maybe empty set of $D_0 = \mathbb{C}\backslash\{0\}$.

6.6. Elastic metamaterial. The last example to be addressed in this paper concerns a model introduced in [36]. It consists of a three-phase composite with coated spheres with radius r_j embedded in a host material represented by a connected bounded open set Ω. Each region of the doubly coated sphere is assumed to be elastic material characterized by mass density ρ_i and Lamé coefficients λ_i and μ_i with the subscript $i = 1, 2, 3$ representing separately the sphere, the coating, and the host. It is shown in [36] that this composite has negative effective mass density ρ_{eff}, negative μ_{eff}, and negative bulk modulus κ_{eff}. We show, using Theorem 5.1, that the elasticity system (5.1) in the presence of this material is well-posed. The effective

Lamé coefficients of the media are defined by

(6.8)
$$\begin{cases} \mu_{eff}(p) = \mu_3 \\ \quad + \dfrac{(\mu_1-\mu_2)r_1^2\left(u'_{r,2}(r_1)+3u'_{\theta,2}(r_1)\right)+(\mu_2-\mu_3)r_1^2\left(u'_{r,2}(r_2)+3u'_{\theta,2}(r_2)\right)}{r_3^2\left(u'_{r,2}(r_3)+3u'_{\theta,2}(r_3)\right)}, \\ \lambda_{eff}(p) = \kappa_{eff}(p) - \dfrac{2}{3}\mu_{eff}(p), \\ \kappa_{eff}(p) = \kappa_3 \\ \quad + \dfrac{r_1(\kappa_1-\kappa_2)E_0^{13}(s,r_1)c_0^{(1)} + r_2(\kappa_2-\kappa_3)\left(E_0^{11}(h,r_2)a_0^{(3)}+E_0^{13}(h,r_2)\right)}{r_3\left(E_0^{11}(h,r_3)a_0^{(3)}+E_0^{13}(h,r_3)\right)}, \end{cases}$$

where E_{ij}^k and $u'_{r,\theta,l}$ depend on p and invoke spherical Bessel and Hankel functions of the first kind (see the appendix of [36]). We check Assumption 10.

(E1) The holomorphy of $\mu_{eff}(p)$ and $\lambda_{eff}(p)$ follows from the holomorphy of spherical Hankel and Bessel functions of the first kind (see [26] for definitions and properties of these special functions). Hence the Lamé coefficients (6.8) are holomorphic on $D_0 = \mathbb{C}\backslash\{0\}$.

(E2) The Lamé coefficients (6.8) do not depend on $x \in \Omega$ giving trivialy the condition.

(E3) According to [36, Figures 4 and 6], for $p_0 = 1$ the coefficients μ_{eff} and λ_{eff} are strictly positive constants.

Thus, Theorem 5.1 shows that for all $p \in D_0\backslash S$, where S is a set of exceptional values, system (5.2)–(5.4)–(6.8) is well-posed.

7. Conclusion and remarks. In this paper we have studied the well-posedness of some linear partial differential equations coming from the modeling of electromagnetics, acoustics, and elastodynamics phenomenons in metamaterials. We have shown some generic well-posedness for each of the systems in the presence of metamaterials, under assumptions relevant for some models from the literature. Moreover, we have successively applied our results on a periodical array of SRR for Maxwell's equations, a chiral metamaterial built from the Ω-particle resonator model, a bi-anisotropic metamaterial, some absorbing boundary conditions of PML type for the wave equation, and a homogenized acoustic metamaterial having negative bulk modulus. We have also examined some elastic metamaterials by introducing a elasticity system for which a well-posedness result has been demonstrated.

However, some remarks have to be formulated. Concerning our results, we do not show that a particular model is well-posed since we discard a discrete, locally finite, and possibly empty set of frequencies. Fortunately the frequencies for which the problem is ill-posed are isolated and the solution is holomorphic. Thus any small variation of p would make it well-posed. On the other hand, we are not able to provide explicit values for the singular frequencies for which the problem is not well-posed since we work with a domain having no specific shape. Nevertheless, we provide in this paper sufficient conditions on the material to ensure discreteness and local finiteness of singular frequencies.

Note also that we require smoothness in the spatial variable (at least Lipschitz continuous) of the multiplicative operators involved in equations studied here through the assumptions of our theorems. Consequently, we cannot study transmission problems between "classical" materials and metamaterials, as is done in [1, 3], since the multiplicative operator $K(p,.)$ is (only) $L^\infty(\overline{\Omega})$.

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