ISSN 1995-4239, Numerical Analysis and Applications, 2012, Vol. 5, No. 2, pp. 116–123. © Pleiades Publishing, Ltd., 2012.
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Scattering of a Scalar Time-Harmonic Wave by N Small Spheres by the Method of Matched Asymptotic Expansions

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Received October 10, 2011

Abstract—In this paper, we construct an asymptotic expansion of a time-harmonic wave scattered by N small spheres. This construction is based on the method of matched asymptotic expansions. Error estimates give a theoretical background to the approach.

DOI: 10.1134/S1995423912020036

Keywords: *Helmholtz equations, matched asymptotic expansions, homogenization.*

1. INTRODUCTION

Scattering of a time-harmonic wave by many small obstacles, also called multiple scattering, has numerous applications in science and engineering. Scattering of acoustic waves by fog or water droplets, electromagnetic waves in composite materials, or pressure waves in aerosols, emulsions, and bubbly liquids constitutes few representative examples of such applications. Another application can also be met in inverse problems such as time reversal [1], where there is a need for efficiently solving the direct scattering problem.

In this paper, we consider the scattering of a scalar plane wave by N small obstacles, included in a bounded domain of $\mathbb{R}^3$. In many cases, the scatterers are spheres, or can be assumed to have such a shape, as for example, when considering dust particles or bubbles. Thus, for simplicity, the scatterers are assumed to be spheres, having the same radius, which is the small parameter destined to tend to zero. Usually, the determination of the wave scattered by N obstacles can be done with multipoles methods [10] requiring the numerical solution of a large linear system. Consequently, such a method does not directly yield the asymptotic expansion of the scattered wave. To obtain this expansion, we use the method of matched asymptotic expansions (cf., e.g., [8, 9, 11, 7, 4]) instead. The main advantage of this procedure is that it yields a convenient way to describe both the field inside a boundary layer enclosing each sphere and the overall behavior of the scattered field. Error estimates ensure a theoretical background to the approach.

2. THE SCATTERING PROBLEM

Let Ω be a bounded open domain of $\mathbb{R}^3$ in which are included N small scatterers. We assume that these small bodies are balls of respective centers $c_j \in \Omega$ of the same radius δ, which is the small parameter destined to tend to zero. These balls are denoted $\mathcal{B}_j^\delta$. This study is dedicated to the scattering of an incident wave $u_{\text{inc}}(x) = e^{i\kappa x \cdot d}$ by these balls and more particularly the asymptotic behavior of the scattered field as δ tends to zero. Symbol d indicates a unit vector of $\mathbb{R}^3$ giving the direction of the incident wave. For simplicity, we focus on the case where the refracting properties of each small scatterer are

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characterized by means of an effective impedance boundary condition, but other instances of scattering problems can be dealt with along the same lines. The total wave u_δ is then the solution to the following boundary-value problem:

$$\begin{cases} (\Delta+\kappa^2)u_\delta(x)=0, \text{ in } \mathbb{R}^3\backslash \bigcup_{j=1}^{N} \overline{\mathcal{B}_j^\delta}, \\ \partial_{\boldsymbol{n}_j} u_\delta = \alpha_j^\delta u_\delta, \text{ on } \mathcal{S}_j^\delta, \quad \lim_{|x|\to+\infty} |x| \left(\partial_{|x|} - i\kappa\right)(u_\delta - u_{\text{inc}}) = 0, \end{cases} \tag{2.1}$$

where α_j^δ is the effective impedance of $\mathcal{B}_j^\delta$ and $\boldsymbol{n}_j$ stands for the unit normal vector on the sphere $\mathcal{S}_j^\delta$ centered on c_j and of radius δ, inwardly directed to $\mathcal{B}_j^\delta$. The investigation of the asymptotic properties of u_δ as δ tends to zero needs to first establish the existence and the uniqueness of u_δ, and its stability with respect to a suitably defined right-hand side for the above system.

The uniqueness is obtained in a straightforward way by well-known arguments based on Rellich's lemma [6] and, more particularly in this case, on the condition

$$\Im m\, \alpha_j^\delta > 0, \ j=1,\ldots,N, \tag{2.2}$$

expressing that the scatterers are made up from energy absorbing materials. For the existence, a standard approach is to introduce a fictitious truncating bounded boundary and to take into account the unbounded part of the domain hence truncated through the related Dirichlet-to-Neumann (DtN) operator (see for instance [2, 3] where this reduction is detailed for similar problems). We just shortly show below how this operator is defined. Let $R>0$ be the radius of a sufficiently large ball $\mathcal{B}_R$ enclosing all of the scatterers $\mathcal{B}_j^\delta$. For any given ϕ in $H^{1/2}(\mathcal{S}_R)$ where $\mathcal{S}_R$ is the sphere limiting $\mathcal{B}_R$, we denote by v the solution to the following boundary-value problem whose existence and uniqueness can be obtained for instance by the limiting-absorption principle (see, e.g., [14]):

$$\begin{cases} v\in \mathcal{D}'\left(\mathbb{R}^3\backslash\overline{\mathcal{B}_R}\right) \text{ such that } \theta v \in H^1\left(\mathbb{R}^3\backslash\overline{\mathcal{B}_R}\right) \ \ \forall \theta\in\mathcal{D}\left(\mathbb{R}^3\right) \\ (\Delta+\kappa^2)v=0 \text{ in } \mathbb{R}^3\backslash\mathcal{B}_R, \\ v=\phi \text{ on } \mathcal{S}_R, \quad \lim_{|x|\to+\infty}|x|\left(\partial_{|x|}-i\kappa\right)v=0. \end{cases} \tag{2.3}$$

The DtN operator $T: H^{1/2}(\mathcal{S}_R)\to H^{-1/2}(\mathcal{S}_R)$ is then defined by $T\phi = -\partial_r v \in H^{-1/2}(\mathcal{S}_R)$.

Let $\Omega_\delta = \mathcal{B}_R\backslash \bigcup_{j=1}^{N}\overline{\mathcal{B}_j^\delta}$. We can then rewrite equation (2.1) on the bounded domain Ω_δ equivalently as follows:

$$\begin{cases} u_\delta\in H^1(\Omega_\delta), \quad (\Delta+\kappa^2)u_\delta = 0 \text{ in } \Omega_\delta, \\ \partial_{\boldsymbol{n}_j}u_\delta = \alpha_j^\delta u_\delta \text{ on } \mathcal{S}_j^\delta, \quad \partial_r u_\delta + Tu_\delta = \partial_r u_{\text{inc}} + Tu_{\text{inc}} \text{ in } \mathcal{S}_R. \end{cases} \tag{2.4}$$

Existence and stability issues for problem (2.4) can be stated in the framework of the following variational formulation:

$$\begin{cases} u_\delta\in H^1(\Omega_\delta), \ \forall v\in H^1(\Omega_\delta), \\ a_\delta(u_\delta,v) = l_\delta(v), \end{cases} \tag{2.5}$$

where $a_\delta(u_\delta,v) = \int_{\Omega_\delta}\langle\nabla u_\delta,\overline{\nabla v}\rangle - \kappa^2 u_\delta\overline{v}\,dx + \int_{\mathcal{S}_R} Tu_\delta\overline{v}d\sigma - \sum_{j=1}^{N}\alpha_j^\delta\int_{\mathcal{S}_j^\delta}u_\delta\overline{v}d\sigma$, and $v\to l_\delta(v)$ is an arbitrary linear form on $H^1(\Omega_\delta)$. We then have the result.

Theorem 2.1. *Under condition* (2.2) *and the following further assumption*

$$\alpha_j^\delta = \nu_j f(\delta),\ f(\delta)>0, \tag{2.6}$$

problem (2.4) *admits a unique solution* u_δ *and the following bound holds true asymptotically for* $\delta \to 0$ *with a constant* C *independent of* δ:

$$\|u_\delta\|_{H^1(\Omega_\delta)} \leq C \sup_{\|v\|_{H^1(\Omega_\delta)} \leq 1} \|l_\delta(v)\| . \tag{2.7}$$

Proof. The existence and uniqueness are obtained in a standard way from a Fredholm alternative. Stability is established by contradiction. We just mention here the two main ingredients of the proof. The first one relies upon the possibility to extend u_δ in a stable way relatively to δ in a function defined inside the $\mathcal{B}_j^\delta$ based on the construction described in [13]: u_δ coincides with a function still denoted by $u_\delta \in H^1(\mathcal{B}_R)$ satisfying

$$\|u_\delta\|_{H^1(\mathcal{B}_R)} \leq C \|u_\delta\|_{H^1(\Omega_\delta)} \tag{2.8}$$

with a constant $C > 0$ independent of δ. The second argument uses the energy absorbing properties of the balls to get that $\lim_{\delta \to 0} f(\delta) \|u_\delta\|_{L^2(\mathcal{S}_j^\delta)} = 0$ when u_δ is a solution of (2.5), uniformly bounded in $H^1(\Omega_\delta)$ relatively to δ, corresponding to a right-hand side such that $\lim_{\delta \to 0} \sup_{\|v\|_{H^1(\Omega_\delta)} \leq 1} \|l_\delta(v)\| = 0.$ □

In the sequel, for simplicity, we assume that the impedance of $\mathcal{S}_j^\delta$ is in the following form $\alpha_j^\delta = \nu_j/\delta$.

3. MATCHED ASYMPTOTIC EXPANSIONS

We now want to write an asymptotic expansion of the solution u_δ to (2.4). However, no expansion does exist that would be simultaneously valid in the proximity of the scatterers and far enough from them. We are thus led to use the method of matched asymptotic expansions to construct an expansion for the field outside the immediate proximity of the balls and another one inside a boundary layer enclosing each of them. Relevant introductions to the method of matched asymptotic expansions can be found, for instance in, [8, 11, 7]. According to the terminology in use for this method, any issue concerning the expansion outside the proximity of the scatterers and inside the boundary layers will be referred to as outer and inner, respectively. We, hence, look for an outer expansion in the form

$$u_\delta(x) = \sum_{k=0}^{n} \delta^k u_k(x) + \underset{\delta \to 0}{o}(\delta^n),\ x \in \mathbb{R}^3 \setminus \bigcup_{j=1}^{N} \{c_j\}. \tag{3.1}$$

The inner expansion corresponds to a zoom on each B_j^δ and is expressed in terms of the fast variables $(R, \Theta) = (|x - c_j|/\delta, (x - c_j)/|x - c_j|)$ as follows:

$$\Pi_\delta^{(j)}(X) = u_\delta\left(\frac{x - c_j}{\delta}\right) = \sum_{k=0}^{n} \delta^k \Pi_k^{(j)}(R, \Theta) + \underset{\delta \to 0}{o}(\delta^n). \tag{3.2}$$

Of course, the coordinates X or (R, Θ) depend on j, but we leave this dependence implicit for simplicity.

The next parts of this section are dedicated to the determination of the above expansions. The outer expansion gives rise to no difficulty: we will show that its coefficients are finite sums of suitable multipoles, that is, linear combinations of products of spherical harmonics by spherical Hankel functions of the first kind. The difficult part actually relies on the determination of the inner expansion, which requires solving a kind of recursive Laplace equation. Similar problems have been considered in $2D$ (see, for instance, [12, 5, 4]). Actually, in these two steps, we show that both inner and outer expansions can be expressed in terms of some unknown constants, which can next be obtained by using matching rules as in [4].

3.1. Outer Expansion

We now derive a general expression for the outer coefficients. Plugging expansion (3.1) into (2.1) and identifying coefficients that correspond to a same power of δ yields

$$\begin{cases} (\Delta+\kappa^2)u_k(x) = 0 \text{ in } \mathbb{R}^3\backslash \bigcup_{j=1}^{N}\{c_j\}, \\ \lim\limits_{|x|\to+\infty} |x| \left(\partial_{|x|} - i\kappa\right)\left(u_k - \delta_k^0 u_{\text{inc}}\right) = 0, \end{cases} \tag{3.3}$$

where $\delta_k^0 = 1$ if $k = 0$ and 0 otherwise. Matching rules given below ensure that u_k has only a finite order of singularity at each c_j, that is, there exists $m_j \in \mathbb{R}$ such that $|x - c_j|^{m_j}|u_k(x)|$ is bounded in some ball $\mathcal{B}_j^{\rho_j}$ centered at c_j and of radius ρ_j.

The following lemma is a crucial step in the characterization of the solutions to (3.3) although its proof is simple.

Lemma 3.1. *Any solution to* (3.3) *with a finite order of singularity at each* c_j *can be expressed as a superposition of multipoles located at each* c_j *as follows*:

$$u_k(x) = \delta_k^o u_{\text{inc}}(x) + \sum_{j=1}^{N}\sum_{n\geq 0} h_n^{(1)}(\kappa|x-c_j|)Y_{n,k}^{(j)}\left(\frac{x-c_j}{|x-c_j|}\right),\ x \in \mathbb{R}^3\backslash\bigcup_{j=1}^{N}\{c_j\}, \tag{3.4}$$

where $h_n^{(1)}$ *is the spherical Hankel function of the first kind of order* n *and* $Y_{n,k}^{(j)}$ *denotes a spherical harmonic of order* n*. Only a finite number of these multipoles are not zero.*

The proof is obtained by a separation of variables around each c_j and by making use of the Rellich Lemma ensuring that the only solution to the Helmholtz equation satisfying the radiation condition in all of $\mathbb{R}^3$ is zero (cf., e.g., [14]). Clearly, the determination of any coefficient of the outer expansion reduces to that of a finite number of constants by appropriately choosing a basis of the spherical harmonics.

3.2. Inner Expansion

Proceeding as for the outer expansion, we plug its expression (3.2) into (2.1) to get that the inner coefficients satisfy the following equation, sometimes referred to as a recursive Laplace equation [5],

$$\Delta\Pi_k^{(j)}(R,\Theta) = -\kappa^2\Pi_{k-2}^{(j)}(R,\Theta),\ \mathbb{R}^3\backslash\overline{\mathcal{B}}, \tag{3.5}$$

where $\mathcal{B}$ stands for the unit sphere of $\mathbb{R}^3$ and $\Pi_k^{(j)}$ satisfies $(\partial_R\Pi_k^{(j)} - \nu_j\Pi_k^{(j)})|_{R=1} = 0$. Such kinds of recursive equation were studied in a two-dimensional setting in [5] but, due to the well-known complications of the space wave equation in even dimensions, this equation can be more easily solved in the three-dimensional case. First, by a separation of variables, we see that it is enough to look for solutions depending on the same spherical harmonic of order n, that is, $\Pi_k^{(j)}(R,\Theta) = \sum\limits_{n\geq 0} P_{n,k}^{(j)}(R)Y_{n,k}^{(j)}(\Theta)$, where $P_{n,k}^{(j)}$ satisfies the following recursive equation:

$$\boldsymbol{L}_n P_{n,k}^{(j)} = \left(\partial_R(R^2\partial_R) - n(n+1)\right)P_{n,k}^{(j)} = -\kappa^2R^2P_{n,k-2}^{(j)}. \tag{3.6}$$

Then, observing that $\boldsymbol{L}_n R^\alpha = (\alpha - n)(\alpha + n + 1)R^\alpha$ and that $P_{n,k}^{(j)}(R) = cR^n + d/R^{n+1}$, for $k = 0, 1$, where c and d are some unknown constants, we easily prove by induction the following result.

Lemma 3.2. *Any solution to the recursive Laplace equation is a finite superposition of particular solutions corresponding to the spherical harmonics* $Y_{n,k}^{(j)}(\Theta)$*, which can be written as follows*:

$$\Pi_k^{(j)}(R,\Theta) = \sum_{n\geq 0}\left\{\sum_{l=0}^{\lfloor k/2\rfloor} c_{l,n,k}^{(j)}R^{-(n+1)+2l} + d_{l,n,k}^{(j)}R^{n+2l}\right\}Y_{n,k}^{(j)}(\Theta),$$

with $2(\lfloor k/2\rfloor + 1)$ *unknown constants and* $\lfloor k/2\rfloor$ *is the largest integer not greater than* $k/2$.

Finally, it is important to note for solving these equations that the first two elements of any solution to the above recursive Laplace equation depend only on a single unknown constant

$$P_{n,k}^{(j)}(R) = \gamma_{n,k}^{(j)} \left\{ (n-\nu_j)R^n + \frac{n+1+\nu_j}{R^{n+1}} \right\}, \quad k = 0, 1. \tag{3.7}$$

3.3. Matching Conditions

The matching conditions link the coefficients of the outer expansion to the inner one and make it possible to uniquely determine both of them. Actually the exact statement of these conditions depends on the particular problem being considered (see, e.g., [12, 4]). For the current scattering problem, we follow the approach of [4]. The matching conditions are then defined as follows successively for $m = 0, 1, 2, \dots$.

- A truncated expansion in the zone far from the small bodies is considered and expressed with respect to the fast variables:

$$u_{m,\delta}(c_j + \delta R\Theta) = \sum_{l=0}^{m} \delta^l u_l(c_j + \delta R\Theta).$$

This function is then expanded in a series of powers of δ and truncated to get:

$$\sum_{l=0}^{m} \delta^l u_l(c_j + \delta R\Theta) = \sum_{l=0}^{m} \delta^l \boldsymbol{U}_{m,l}^{(j)}(R,\Theta) + \underset{\delta\to 0}{o}(\delta^m). \tag{3.8}$$

- The matching conditions are then defined as follows:

$$\boldsymbol{U}_{m,k}^{(j)} - \Pi_k^{(j)} = \underset{R\to+\infty}{o}\left(\frac{1}{R^{m-k}}\right), \quad \forall j = 1, \dots, N. \tag{3.9}$$

It is worth noting that assuming a series expansion for $u_{m,\delta}(c_j + \delta R\Theta)$ in the form (3.8) implies that the order of singularity of the outer coefficients at each c_j is finite.

To conclude this part, we note that the above matching rules, together with Lemmas 3.1 and 3.2, establish that these expansions do exist and can be uniquely determined at any order. The proof of this claim actually reduces to checking that the intermediate functions $\boldsymbol{U}_{m,k}^{(j)}$ can be effectively obtained. This is actually achieved using the translation formulas for multipoles and using the Gegenbauer formula to write the spherical harmonic expansion of the incident field. All the relevant formulas can be found in [10].

3.4. Explicit Determination of the First Few Terms of the Asymptotic Expansions

We now determine the outer and the inner expansions up to order 1. One ingredient is furnished by the expansion for the small values of the argument of the spherical Hankel functions [10]:

$$h_n^{(1)}(z) = (-i)^{n+1}\frac{e^{iz}}{z}\sum_{l=0}^{n}\frac{i^l(l+n)!}{l!(2z)^l(n-l)!} = \sum_{l\geq -(l+1)} h_{l,n} z^l, \tag{3.10}$$

from which we infer that $h_n(z) \sim_{z\to 0} C/z^{n+1}$.

Now, since u_0 must remain bounded as $x \to c_j$, Lemma 3.1 implies that $u_0 = u_{\text{inc}}$. Taylor expansion at order 0 then gives that $\boldsymbol{U}_{0,0}^{(j)} = u_{\text{inc}}(c_j)$. Matching rules (3.9) and the general expression (3.7) for the solution to recursive Laplace equation then yield $\Pi_0^{(j)}(R,\Theta) = \left(1 - \frac{\nu_j}{1+\nu_j}\frac{1}{R}\right) u_{\text{inc}}(c_j)$, hence, completing the determination of the zero-order asymptotic expansions.

Proceeding as for u_0, we use Lemma 3.1 to get that u_1 is determined up to N constants $u_1(x) = \sum_{j=1}^{N} h_0^{(1)}(\kappa|x-c_l|)Y_{0,1}^{(j)}$, since spherical harmonics of order zero reduce to constants. The matching functions are next determined from a Taylor expansion and above formula (3.10):

$$\boldsymbol{U}_{1,0}^{(j)}(R,\Theta) = u_{\mathrm{inc}}(c_j) - \frac{i}{\kappa R}Y_{0,1}^{(j)},$$
$$\boldsymbol{U}_{1,1}^{(j)}(R,\Theta) = R\nabla u_{\mathrm{inc}}(c_j)\cdot\Theta + Y_{0,1}^{(j)} + \sum_{l\neq j} h_0^{(1)}(\kappa|c_j-c_l|)Y_{0,1}^{(l)}.$$

Matching rules and general first terms expressions for the solution to Laplace recursive equation (3.7) give the asymptotic expansions at order 1:

$$u_1(x) = \frac{\kappa}{i}\sum_{l=1}^{N} u_{\mathrm{inc}}(c_l)\frac{\nu_l}{1+\nu_l}h_0^{(1)}(|x-c_l|),$$

$$\Pi_1^{(j)}(R,\Theta) = \left(1-\frac{\nu_j}{1+\nu_j}\frac{1}{R}\right)\left(\frac{\kappa}{i}u_{\mathrm{inc}}(c_j)\frac{\nu_j}{1+\nu_j} + \sum_{l=1,\, l\neq j}^{N} h_0^{(1)}(\kappa|c_j-c_l|)\frac{\kappa}{i}u_{\mathrm{inc}}(c_l)\frac{\nu_l}{1+\nu_l}\right) + \left(R+\frac{1-\nu_j}{2+\nu_j}\frac{1}{R^2}\right)(\nabla u_{\mathrm{inc}}(c_j)\cdot\Theta).$$

Similar calculations, which are not detailed here, make it possible to determine the asymptotic expansions at any order. It is worth noting, however, that Taylor's expansions can no more be used for recovering the expressions of the intermediate functions $\boldsymbol{U}_{m,\ell}^{(j)}$. The involved functions must be expanded in terms of spherical harmonics to be able to solve the corresponding recursive Laplace equations. This can be achieved by using instead the translation formulas for multipoles as they are stated in [10], for instance.

4. ERROR ESTIMATES

In this part, we derive error estimates that give a rigorous justification to the above formal asymptotic expansions.

4.1. The Uniformly Valid Approximation

Neither the outer nor the inner asymptotic expansion can be used to approximate u_δ everywhere. However, a uniformly valid approximation can be built by suitably mixing both of them. Let $\chi \in \mathcal{D}(\mathbb{R}^+)$ be a cut-off function such that $\chi(r) = 1$ for $r \leq 1$ and $\chi(r) = 0$ for $r = 2$. A uniformly valid approximation $\widetilde{u}_{\delta,n}$ can then be defined by

$$\widetilde{u}_{\delta,n}(x) = \left(1-\sum_{j=1}^{N}\chi_\delta^{(j)}(x)\right)\sum_{k=0}^{n}\delta^k u_k(x) + \sum_{j=1}^{N}\chi_\delta^{(j)}(x)\sum_{k=0}^{n}\delta^k\Pi_k^{(j)}\left(\frac{|x-c_j|}{\delta},\Theta\right), \tag{4.1}$$

where $\chi_\delta^{(j)}(x) = \chi\left(|x-c_j|/\sqrt{\delta}\right)$. Remark that the uniformly valid approximation is equal to the outer expansion away from the small scatterers and to the inner expansion in an "asymptotic" neighborhood of the small balls. More precisely, we have $\widetilde{u}_{\delta,n}(x) = \sum_{k=0}^{n}\delta^k u_k(x)$, for x such that $\max_{j=1,\ldots,N}|x-c_j| \geq 2\sqrt{\delta}$, and $\widetilde{u}_{\delta,n}(x) = \sum_{k=0}^{n}\delta^k\Pi_k^{(j)}\left(\frac{|x-c_j|}{\delta},\Theta\right)$, $\delta < |x-c_j| \leq \sqrt{\delta}$. These two expansions are not zero simultaneously only in the matching zones $\mathcal{M}_\delta^{(j)} = \left\{x\in\Omega_\delta |\, \sqrt{\delta}\leq|x-c_j|\leq 2\sqrt{\delta}\right\}$.

4.2. Residual Estimate and Stability Property

A quite natural manner for getting a bound on the error resulting from the approximation of u_δ by $\widetilde{u}_{\delta,n}$ is to use stability estimate (2.1) and to suitably bound the residual. If $e_{\delta,n} = u_\delta - \widetilde{u}_{\delta,n}$, plugging $e_{\delta,n}$ into system (2.4), we get

$$\begin{cases} (\Delta + \kappa^2) e_{\delta,n} = (\Delta + \kappa^2)\widetilde{u}_{\delta,n} \text{ in } \Omega_\delta, \\ \partial_{\boldsymbol{n}_j} e_{\delta,n} = \alpha_j^\delta e_{\delta,n}, \text{ on } \mathcal{S}_j^\delta, \quad \partial_r e_{\delta,n} + T e_{\delta,n} = 0 \text{ on } \mathcal{S}_R. \end{cases}$$

Stability property (2.1) then shows that the following bound holds true:

$$\|e_{\delta,n}\|_{H^1(\Omega_\delta)} \le C \left\|(\Delta + \kappa^2)\widetilde{u}_{\delta,n}\right\|_{L^2(\Omega_\delta)}$$

with C independent of δ. Long but straightforward estimates readily yield then the following bound.

Theorem 4.1. *The following estimates holds true asymptotically for* $\delta \to 0$*:*

$$\|u_\delta - \widetilde{u}_{\delta,n}\|_{H^1(\Omega_\delta)} \le C N \delta^{\frac{n}{2}+\frac{1}{4}},$$

where $C > 0$ *is a constant that does not depend on* δ *or on the number* N *of small balls.*

Now let $\rho > 0$ be sufficiently small such that $\mathcal{B}_j^\rho$ does not contain any c_ℓ except c_j. Define $\mathcal{F}_\rho$ as $\Omega_\delta \backslash \bigcup_{j=1}^{j=N} \overline{\mathcal{B}_j^\rho}$. As a corollary, we get the following bound from the general above estimate and by making use of a simple triangular inequality:

$$\|u_\delta - u_{\delta,1}\|_{H^1(\mathcal{F}_\delta)} \le C\delta^2.$$

Since $u_{\delta,1}(x) = u_{\text{inc}}(x) + \delta \frac{\kappa}{i} \sum_{l=1}^{N} u_{\text{inc}}(c_l) \frac{\nu_l}{1+\nu_l} h_0^{(1)}(|x - c_l|)$ for $x \in \mathcal{F}_\rho$, it appears that a first-order approximation of the field scattered by N small balls is nothing else but the field obtained by neglecting the mutual interactions between the small scatterers.

ACKNOWLEDGMENTS

This work was supported by the French Research Agency, project ANR-08-SYSC-001.

REFERENCES

1. Antoine, X., Pincon, B., Ramdani, K., and Thierry, B., Far-Field Modelling of Electromagnetic Time-Reversal and Application to Selective Focusing on Small Scatterers, *SIAM J. Appl. Math.*, 2008, vol. 69, no. 3, pp. 830–844.
2. Bendali, A. and Lemrabet, K., The Effect of a Thin Coating on the Scattering of a Time-Harmonic Wave for the Helmholtz Equation, *SIAM J. Appl. Math.*, 1996, vol. 6, no. 5, pp. 1664–1693.
3. Bendali, A. and Lemrabet, K., Asymptotic Analysis of the Scattering of a Time-Harmonic Wave by a Perfectly Conducting Metal Coated with a Thin Dielectric Shell, *Asympt. Anal.*, 2008, vol. 57, pp. 199–227.
4. Bendali, A., Makhlouf, A., and Tordeux, S., Justification of the Cavity Model in the Numerical Simulation of Patch Antennas by the Method of Matched Asymptotic Expansions, *SIAM Multiscale Model. Simulat.*, 2010, no. 8, pp. 1902–1922.
5. Claeys, X., Haddar, H., and Joly, P., Etude d'un Problème Modèle pour la Diffraction par des Fils Minces par dÉveloppements Asymptotiques Raccordés cas 2d, Research Report RR-5839, INRIA Rocquencourt, 2006.
6. Colton, D. and Kress, R., *Inverse Acoustic and Electromagnetic Scattering Theory, Ser. Appl. Math.*, New York: Springer-Verlag, 1992, no. 93.
7. Cousteix, J. and Mauss, J., *Asymptotic Analysis and Boundary Layers*, New York: Springer-Verlag, 2007.
8. Dyke, M.V., *Perturbation Methods in Fluid Mechanics*, Stanford, California: The Parabolic Press, 1975.
9. Eckhaus, W., *Matched Asymptotic Expansions and Singular Perturbations, North-Holland Math. Stud.*, Amsterdam: North-Holland Publishing, 1973, no. 6.

10. Gumerov, N.A. and Duraiswamy, R., *Fast Multipole Method for the Helmholtz Equation in Three Dimensions*, Amsterdam: Elsevier, 2004.
11. Il'in, A.M., *Matching of Asymptotic Expansions of Solutions of Boundary-Value Problems*, Providence: Am. Math. Soc., 1992.
12. Joly, P. and Tordeux, S., Matching of Asymptotic Expansions for Wave Propagation in Media with Thin Slots I: The Asymptotic Expansion, *SIAM Multiscale Model. Simulat.*, 2006, vol. 5, no. 1, pp. 304–336.
13. Maz'ya, V.G. and Poborchi, S.V., Extension of Functions in Sobolev Spaces on Parameter Dependent Domains, *Math. Nachrichten*, 1996, no. 178, pp. 5–41.
14. Wilcox, C.H., *Scattering Theory for the d'Alembert Equation in Exterior Domains*, Berlin: Springer-Verlag, 1975, no. 442.