Mathematical study of many example of homogenized metamaterials

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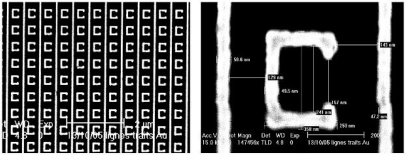
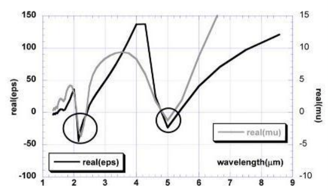
Framework :

- Artificial medium created by homogenization of small physical components.
- Exotic behavior at some frequencies.



(a) Periodic array of SRR



(b) Real part of homogenized parameters

Fig.: Real part of ε and μ of a periodic array of SRR,
Kanté B., A. De Lustrac et als, Metamaterials for optical and radio communications.

Some application of metamaterials :

- Super-Lens (with materials with negative refractive index).
- Alice's mirror,
- Control of light (with photonic crystals).
- Invisibility, cloaking
- Seismic protection for buildings.
- ...

Setting of the problem

Statement of the problem

Hypothesis :

- $\Omega \subset \mathbb{R}^3$ = a bounded open set of $\mathbb{R}^3$ with $\mathcal{C}^2$ boundary with outward unitary normal denoted by $\mathbf{n}$.
- Metamaterials = compact material.

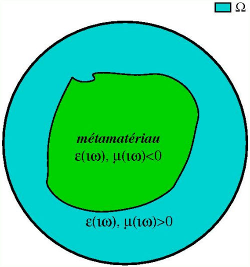


Fig.: Geometric setting of the problem

Maxwell's system in Laplace transform :

$$\left\{ \begin{array}{l} \textit{Find } (E,H) \in \mathcal{D}(\mathbb{M}) \textit{ such that :} \\ (K(p,x) + \mathbb{M}) \begin{pmatrix} E \\ H \end{pmatrix} = f, \;\; L^2(\Omega)^6 \end{array} \right.$$

where :

$$\mathcal{D}(\mathbb{M}) = \left\{ (E,H) \in H(curl,\Omega)^2 \mid \mathbf{n}(x) \times (E + \Lambda(x)(\mathbf{n}(x) \times H)) = 0, \;\; H^{-1/2}(\partial\Omega) \right\}$$

$$p = iw + \eta, \;\; \mathbb{M} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}$$

and :

$$K(p,x) = \begin{pmatrix} p\varepsilon(p,x) & 0 \times \mathbb{I}_3 \\ 0 \times \mathbb{I}_3 & p\mu(p,x) \end{pmatrix} \in Hom(\mathbb{C}^6),$$

$$\forall u \in \mathbb{C}^3, Re\,\langle (\Lambda + \Lambda^*)u, \overline{u} \rangle \geq \alpha |u|^2.$$

Main problem

Physical parameters of metamaterials depends on the frequency and could become negative definite on the behavior of some p.
$\Longrightarrow$ The multiplicative operator $K(p)$ is no longer coercive for some p.

Goal

Give conditions on the homogenized parameters of the metamaterials leading to the well-posedness of the Maxwell's system.

An example : periodic array of SRR

B. Kanté, SN Burokur, F. Gadot, and A. de Lustrac. Métamatériau à indice de réfraction négatif en infrarouge.

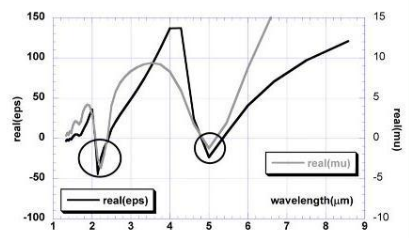


Fig.: Periodic array of S.R.R

Homogenized parameters

$$[\varepsilon(p,x)] = \left(1 + \frac{\omega_p^2}{p^2}\right)\mathbb{I}_3, \quad [\mu(p,x)] = \left(1 + \frac{\delta p^2}{-p^2 - \omega_0^2 + p\Gamma}\right)\mathbb{I}_3.$$

Main result

Assumptions on the metamaterial :

- (p-Regularity) $p \in D_0 \longrightarrow K(p,x)$ is holomorphic for almost all $x \in \Omega$.
- (x-regularity) $x \in \overline{\Omega} \longrightarrow K(p,x)$ belong to $L^\infty(\overline{\Omega})$ for all $p \in D_0$.
- (Classical) There exist $p_0 \in D_0$ such that $Re\,(K(p_0) + K(p_0)^*) \succeq \alpha$.
- (technical...) There exist $a \in Lip(\overline{\Omega}, \mathcal{O}(D_0))$ such that $Re\,(a(p)K(p) + a(p)^* K(p)^*) \succeq \alpha$.

Theorem

Assume that (p-Regularity)-(x-Regularity)-(Classical)-(technical...) are satisfied. Then the Maxwell's system in presence of metamaterial is well-posed for all $p \in D_0 \backslash S$ where S is a discrete set of D_0.

Idea beyond the proof :

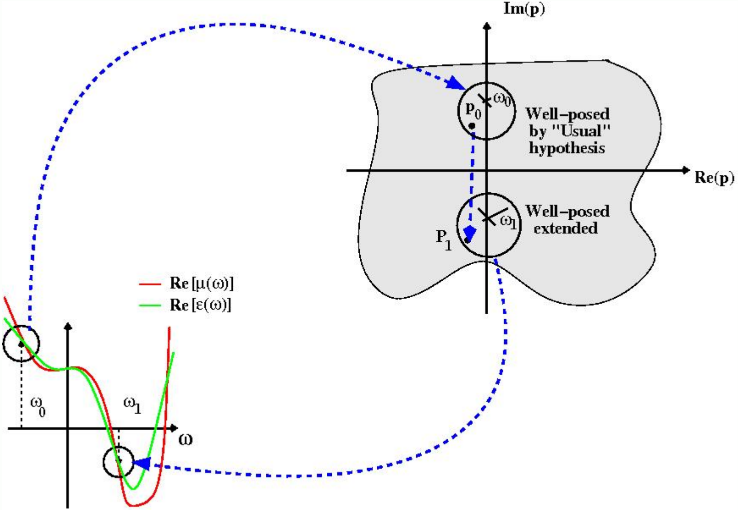


Fig.: Idea of proof

Proof part 1

Steady point...

Goal :

Solve the following equation : $\begin{cases} \text{Find } (E,H) \in \mathcal{D}(\mathbb{M}) \text{ such that :} \\ (K(p,x) + \mathbb{M}) \begin{pmatrix} E \\ H \end{pmatrix} = f, \; L^2(\Omega)^6 \end{cases}$ where

$\mathcal{D}(\mathbb{M}) = \{(E,H) \in H(curl,\Omega)^2 \; + \; B.C.\}$ and $\mathbb{M} = \begin{pmatrix} 0 & -\nabla\times \\ \nabla\times & 0 \end{pmatrix}$.

A remark :

- The embedding $\mathcal{D}(\mathbb{M}) \subset L^2(\Omega)^6$ is not compact.
  $\Longrightarrow$ Maxwell operator $(\mathbb{M}, \mathcal{D}(\mathbb{M}))$ has no compact resolvent.

Consequence

Fredholm analytical theory not applicable yet.

An idea to recover some compactness :

- We have the inequality (Majda, Poincaré-Friedrich)
  $\forall u \in \mathcal{D}(\mathbb{M}) \cap H(div, \Omega)^2$:

  $$\| u \|_{(H^1(\Omega))^6} \leq C \left\{ \| u \|_{(L^2(\Omega))^6} + \| \mathbb{M}u \|_{(L^2(\Omega))^6} + \| \mathbb{Q}u \|_{(L^2(\Omega))^6} \right\},$$

  with $\mathbb{Q} = \begin{pmatrix} div & 0 \\ 0 & div \end{pmatrix}$.
  $\Longrightarrow$ One need to control the divergence of the fields to get some compactness.

Main tool : The Hodge decomposition

$$L^2(\Omega)^3 = \nabla \left(H_0^1(\Omega) \right) \oplus H(div0, \Omega).$$

Remark : For simplicity, we assume that Ω is simply connected.

Some notations :

- $P_\nabla : L^2(\Omega)^6 \longrightarrow \left(\nabla\left(H_0^1(\Omega)\right)\right)^2$, $P_0 : L^2(\Omega)^6 \longrightarrow (H(div0,\Omega))^2$.

Equivalent system :

$$\begin{cases} Find\ \ u = P_0 u + P_\nabla u \in \mathcal{D}(\mathbb{M})\ such\ that : \\ P_\nabla K(p,.)P_\nabla u = f - P_\nabla K(p,.)P_0 u, \\ P_0 K(p,.)P_0 u + P_0 \mathbb{M} P_0 u + P_0 K(p,.)P_\nabla u = P_0 f. \end{cases}$$

What have we won ?

The unbounded operator $\left(P_0\mathbb{M}P_0, \mathcal{D}(\mathbb{M}) \cap H(div0,\Omega)^2\right)$ has compact resolvent.

Lemma

The operator $P_\nabla K(p) P_\nabla \in \mathcal{B}\left(\left[(\nabla H_0^1(\Omega)\right]^2\right)$ has bounded-holomorphic inverse for all $p \in D_0 \backslash S$.

Proof.

- Since $P_\nabla u = [\nabla\phi_1, \nabla\phi_2]$, one has to solve the boundary problem :

$$\begin{cases} \text{Find } \phi \in H_0^1(\Omega)^2 \text{ such that :} \\ \mathbb{Q}(K(p)\,[\nabla\phi_1, \nabla\phi_2]) = h, \ \ H^{-1}(\Omega)^2. \end{cases}$$

- Setting $\psi = a(p)\phi$, one gets :

$$\begin{cases} \text{Find } \psi \in H_0^1(\Omega)^2 \text{ such that :} \\ \mathbb{Q}(K(p)a(p)\,[\nabla\psi_1, \nabla\psi_2]) + \mathbb{Q}\left(K(p)[\nabla a(p), \nabla a(p)]\psi\right) = h, \ \ H^{-1}(\Omega)^2. \end{cases}$$

- Problems well-posed for $p = p_0$ by (Classical).

■

Proof part 4 (bis)

Solving the elliptic part (bis)

- From Lax-Milgram lemma and assumption (technical...), the operator

$$\mathbb{T}(p) = \{\mathbb{Q}(K(p)a(p)[\nabla.,\nabla.])\}^{-1} : H^{-1}(\Omega)^2 \longrightarrow L^2(\Omega)^2$$

  is compact and holomorphic.

- One need then to solve :

$$\left\{ \begin{array}{c} \textit{Find } \psi \in L^2(\Omega)^2 \textit{ such that :} \\ (\mathbb{I} + \mathbb{T}(p)\{\mathbb{Q}K(p)[\nabla a(p), \nabla a(p)]\})\,\psi = \mathbb{T}(p)h\, H^{-1}(\Omega)^2. \end{array} \right.$$

- The application $p \in D_0 \longrightarrow \mathbb{T}(p)\{\mathbb{Q}K(p)[\nabla a(p), \nabla a(p)]\} \in \mathcal{B}_0(L^2(\Omega)^2)$ is holomorphic.

Conclusion :

Problem well-posed for all $p \in D_0 \backslash S$ by the analytical Fredholm theory.

- One has $P_\nabla u = (P_\nabla K(p) P_\nabla)^{-1} [P_\nabla f - P_\nabla K(p) P_0 u]$.
- To get $P_0 u$, one need to invert $\forall p \in D_0 \backslash S$ the holomorphic family of closed operators with compact resolvent :

$$\left(\mathbb{B}(p) + P_0 \mathbb{M} P_0, \mathcal{D}(\mathbb{M}) \cap H(div0, \Omega)^2\right),$$

$$\mathbb{B}(p) = P_0 K(p) P_0 - P_0 K(p) P_\nabla (P_\nabla K(p) P_\nabla)^{-1} P_\nabla K(p) P_0 \in \mathcal{B}\left(H(div0, \Omega)^2\right.$$

- From assumption (Classical), the operator $\mathbb{B}(p_0) + P_0 \mathbb{M} P_0$ is invertible.

Conclusion :

From Fredholm analytical theory, the holomorphic family of operators $\mathbb{B}(p) + P_0 \mathbb{M} P_0$ is invertible for all $p \in D_0 \backslash S$, where S is a discrete set.

Some applications

Maxwell's equations with dielectric materials

Parameters of a dielectric material

$[\varepsilon(p,x)] = \epsilon(x)\mathbb{I}_3,\ [\mu(p,x)] = \mu(x)\mathbb{I}_3,$
$\epsilon,\ \mu \in L^\infty(\overline{\Omega})$ and $\epsilon(x) \geq \alpha > 0,\ \mu(x) \geq \alpha > 0$ for almost all $x \in \Omega$.

- (x-regularity)-(p-regularity) satisfied with $D_0 = \mathbb{C}$.
- (''Classical'') checked with $p_0 = 1$.
- (technical...) verified with $a(p,x) = \frac{1}{p}$.

Conclusion

Maxwell's equation with dielectric materials well-posed for all $p \in D_0 \backslash (S \cup \{\ 0\})$.

Periodic array of Split-Ring-Resonator (1)

B. Kanté, S.N. Burokur, F. Gadot, and A. de Lustrac. Métamatériau à indice de réfraction négatif en infrarouge.

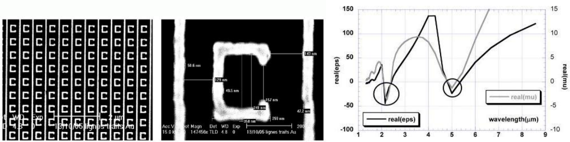


Fig.: Réseau périodique de S.R.R

Homogenized parameters

$$[\varepsilon(p,x)] = \left(1+\frac{\omega_p^2}{p^2}\right)\mathbb{I}_3,\ [\mu(p,x)] = \left(1+\frac{\delta p^2}{-p^2-\omega_0^2+p\Gamma}\right)\mathbb{I}_3, \text{ for } x\in\Theta\subset\Omega$$

$[\varepsilon(p,x)] = [\mu(p,x)] = 1$ for $x \in \overline{\Omega}\backslash\Theta$.

$+$ smooth transition between 1 and the parameter of the *SRR*.

$\Longrightarrow [\varepsilon(p,.)], [\mu(p,.)] \in \mathcal{C}^1(\Omega)$.

- (x-regularity)-(p-regularity) checked for

$$D_0 = \mathbb{C}\backslash\left\{\frac{\Gamma-\sqrt{\Gamma^2-4\omega_0^2}}{2}, 0, \frac{\Gamma+\sqrt{\Gamma^2-4\omega_0^2}}{2}\right\}.$$

- Taking $p_0 = \frac{\Gamma}{2} > 0$. The function $f : p \in \mathbb{R}\backslash\{0\} \longrightarrow -p^2 - \omega_0^2 + p\Gamma \in \mathbb{R}$ reach its maximum for p_0 and $f(p_0) > 0$.
  $\Rightarrow K_0(p_0) = p_0 Diag([\varepsilon], [\mu])$ is coercive $\Rightarrow$ ("Classical") is checked.
- Taking $a(p, x) = \dfrac{1}{p[\varepsilon(p, x)]} \in \mathcal{C}^1(\Omega) \Rightarrow$ (technical...) is confimed.

Conclusion

Maxwell's equation with homogenized smooth $S.R.R$ are well-posed for all $p \in D_0\backslash S$.

Numerical simulation

Remarks

- Well-posedness extended from $p = p_0$ to other p under the assumptions :
  - (p-Regularity) $p \in D_0 \longrightarrow K(p,x)$ is holomorphic for almost all $x \in \Omega$.
  - (x-regularity) $x \in \overline{\Omega} \longrightarrow K(p,x)$ belong to $L^\infty(\overline{\Omega})$ for all $p \in D_0$.
  - (Classical) There exist $p_0 \in D_0$ such that $Re\left(K(p_0) + K(p_0)^*\right) \succeq \alpha$.
  - (technical...) There exist $a \in Lip(\overline{\Omega}, \mathcal{O}(D_0))$ such that $Re\left(a(p)K(p) + a(p)^* K(p)^*\right) \succeq \alpha$.
- Maxwell's equation well-posed even for non-coercive $K(p)$.
- The set of exceptional frequencies S is not known.
  $\Longrightarrow$ We can't say whether the well-posedness holds or not for specific p.
- The solution is continuous with respect to the data :

$$\left\{ \int_\Omega |E(p,x)|^2 + |H(p,x)|^2 dx \right\}^{\frac{1}{2}} \leq C(p) \left\{ \int_\Omega |f|^2 \, dx \right\}^{\frac{1}{2}}, \ \forall p \in D_0 \backslash S.$$

A numerical scheme for metamaterials

Idea : From [A.K. Aziz, S. Leventhal, Finite element approximation for first order systems, SIAM J Num An, 1978.] $\Rightarrow$ Limited to coercive operators.

Variational formulation :

$$\textit{Find } u_h = (E_h, H_h) \in \mathcal{V}_h \subset \mathcal{D}(\mathbb{M}) \textit{ such that } \forall \psi_h \in (K(p,x) + \mathbb{M})\, \mathcal{V}_h :$$

$$\int_\Omega \langle (K(p,x) + \mathbb{M})\, u_h, \overline{\psi_h} \rangle \, dx = \int_\Omega \langle f, \overline{\psi_h} \rangle \, dx$$

Theorem

Assume that $\forall u \in H^s(\Omega) \cap \mathcal{D}(\mathbb{M})$ *there exists* $\widetilde{u}_h \in \mathcal{V}_h$ *such that*

$$\| u - \widetilde{u}_h \|_{\mathcal{D}(\mathbb{M})} \leq C h^s \| u \|_{H^s(\Omega) \cap \mathcal{D}(\mathbb{M})} .$$

Then if the solution $u \in H^s(\Omega) \cap \mathcal{D}(\mathbb{M})$ *one has the error inequality :*

$$\| u - u_h \|_{L^2(\Omega)^6} \leq C h^s \| u \|_{H^s(\Omega) \cap \mathcal{D}(\mathbb{M})} .$$

- $\Omega =]a, b[\subset \mathbb{R}$.

$$\left\{ \begin{array}{l} Find\ (u, v) \in H_0^1(\Omega) \times H^1(\Omega)\ such\ that : \\ \quad p\varepsilon(p, x)u - \partial_x v = f_1, \\ \quad p\mu(p, x)v - \partial_x u = f_2 \end{array} \right. \tag{1}$$

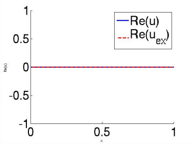
Remark :

Problem well-posed if (x-p-regularity) and ("Classical") hold.

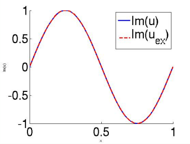
An exact solution :

- $\Omega =]0, 1[,\ f_1(x) = ip\varepsilon \sin(2\pi x) + \sin(x), f_2(x) = p\mu \cos(x) - 2i\pi \cos(2\pi x)$.
- $u(x) = i \sin(2\pi x),\ v(x) = \cos(x)$.
- $p = iw,\ w = 15$.

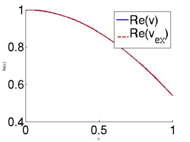
Numerical experiments (1) : the vaccum



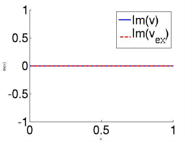
(a) Re(u)



(b) Im(u)

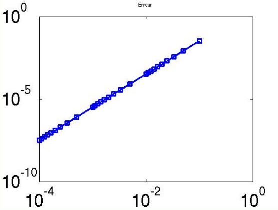


(c) Re(v)

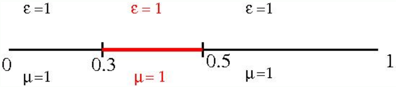


(d) Im(v)

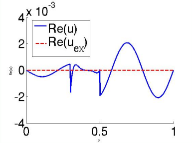
Fig.: The vacuum



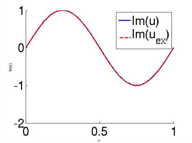
(a) Error, $\| u - u_h \|_{L^2}$



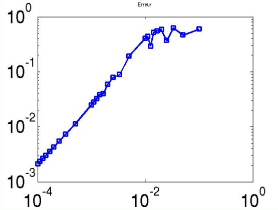
(b) Geometrie



(c) Re(u)



(d) Im(u)



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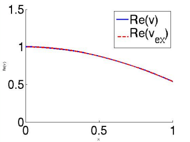
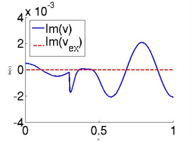
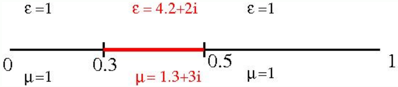


(e) Re(v)



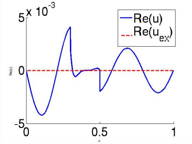
(f) Im(v)



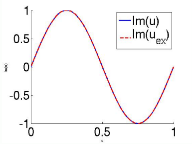
(b) Geometrie

Fig.: A dielectric media

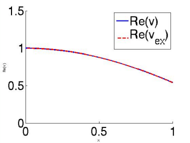
Numerical experiments (3) : A metamaterial



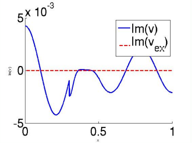
(c) Re(u)



(d) Im(u)

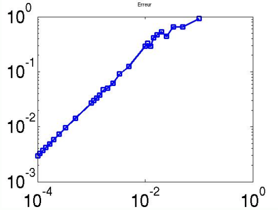


(e) Re(v)

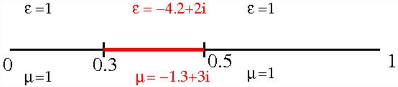


(f) Im(v)

Fig.: A metamaterial



(a) Error, $\| u - u_h \|_{L^2}$



(b) Geometrie

Conclusions and prospects

Conclusion :

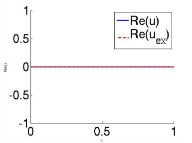
- smooth metamaterials $\Longrightarrow$ well-posed problems.
- Numerical approximation $\Longrightarrow$ OK with some kind of finite element method.

Prospects :

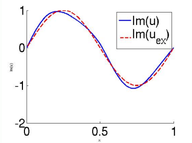
- Study of the well-posedness for L^∞ metamaterials (transmission problems ? ? ?).
- Convergence of Discontinuous Galerkin methods.
- Numerical test in 2D.
- ....

Thank you for you attention.

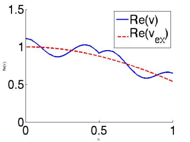
Numerical experiments : Metamaterial out of the topic.



(c) Re(u)



(d) Im(u)



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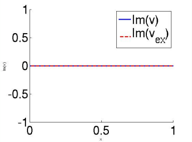
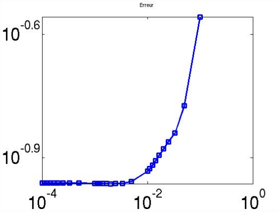


(f) Im(v)

Fig.: A dielectric media



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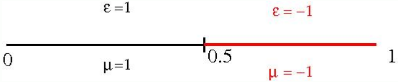

(b) Geometrie