Theoritical Analysis of Some Homogenized Metamaterials and Application of PML to Perform Cloaking and Back-scattering Invisibility

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Abstract— A general Drude-Born-Fedorov (DBF) system, with materials like metamaterials, on a bounded domain is studied. Due to the presence, for some pulsation, of negative permittivity and permeability, the well-posedness of this kind of system remains a steady point. In this paper, we treat examples dealing with homogenized metamaterials like a periodic array of Split-Ring-Resonators (SRR), homogenized dielectric-chiral photonic crystals and convex absorbing boundary conditions of PML type have been treated. At last, we show both theoretically and numerically that PML can perform electromagnetic cloaking and back-scattering invisibility.

1. INTRODUCTION

In 1968, V. G. Veselago [11] theoretically studied electromagnetic phenomenons in materials with simultaneous negative permittivity $[\varepsilon]$ and permeability $[\mu]$. Due to the reversed Poynting vector, he named these materials "Left-Handed-Materials" (LHM). He also remarks that these materials had exotics properties (reversed Doppler effect for example).

The keen interest for the LHM was initiated by J. B. Pendry's in 2000 [6]. He managed to build LHM using a periodic array of Split-Ring-Resonators (SRR) and suggested making a perfect lens with these structures. Futhermore, these exotic building were named "metamaterials".

However, as metamaterials can't be found in nature as negative indexes materials, they are usually the result of an homogenization (see for example [3, 4, 6, 7, 9]). The problem arises from the choice for homogenization's method. Every method can lead to a different homogenized material and by the way to a new partial diffential equation. The question is then to know, in a first step, which of these homogenized problems are well-posed. A second step will be linked with a suitable approximation of the solution of these homogenized systems. Unfortunatly, the well-posedness of these homogenized problems remains a steady point. In fact, homogenized metamaterials have their permitivitty and permeability depending on the pulsation w and start to be negative definite for some w. This implies that the well-posedness of the DBF system, which establishes electromagnetic phenomenons in chiral materials, can't be a priori issued.

So, we are going to study a generalized DBF system including some negative index phenomenons. A well-posedness result for this generalised DBF system is given in the first section. Next, homogenized dielectric-chiral photonic crystals [4], periodical arrangement of Split-Ring-Resonator (SRR) [7] and convex absorbing boundary conditions of PML type for Maxwell's equations [5], seen as "ideal metamaterial", are studied. At last, we show theoretically and check numerically that convex PML for Maxwell's equations can perform electromagnetic cloaking and back-scattering invisibility.

2. MATHEMATICAL FRAMEWORK

Lots of electromagnetic phenomenons in homogenized chiral photonic crystals can be modelized by the following Drude-Born-Fedorov (DBF) system (1), (2) (e.g., [3]):

$$\left\{\begin{array}{l}\text{Find } (e,h)\in H(curl,\Omega)^2 \text{ such that :}\\ (K_0(p,x)+K_1(p,x)\mathbb{M})\begin{pmatrix} e\\ h\end{pmatrix} = f,\quad x\in\Omega\\ \mathbf{n}(x)\times(e+\Lambda(x)\mathbf{n}(x)\times h)=0,\quad x\in\partial\Omega\ ,\end{array}\right. \tag{1}$$

$$\begin{array}{l}\mathbb{M}=\begin{pmatrix} 0 & -\nabla\times\\ \nabla\times & 0\end{pmatrix},\quad K_0(p,x)=\begin{pmatrix} p[\varepsilon(p,x)] & 0\\ 0 & p[\mu(p,x)]\end{pmatrix},\\ K_1(p,x)=\begin{pmatrix}\mathbb{I}_3 & p[\beta(p,x)][\varepsilon(p,x)]\\ -p[\beta(p,x)][\mu(p,x)] & \mathbb{I}_3\end{pmatrix}.\end{array} \tag{2}$$

where e and h are the electric and the magnetic fields. The permitivity $[\varepsilon]$, the permeability $[\mu]$ and the chirality $[\beta]$ are the homogenized parameters of the media which are smoothly depending both on $p = iw$, w is the pulsation, and x, $f \in L^2(\Omega)^6$. At last Ω is a simply connected bounded domain in $\mathbb{R}^3$ with C^1 boundary whose outward unitary normal is denoted $\mathbf{n}$ and $\Lambda \in Lip(\Omega, \mathcal{L}(\mathbb{C}^6))$ such that $\text{Re}(\Lambda(x)+\Lambda^*(x))$ is nonnegative. We also set $\mathbf{H} = \{(e, h)^T \in (L^2(\Omega)^3)^2,\ (\nabla\times e, \nabla\times h)^T \in (L^2(\Omega)^3)^2$ and $\mathbf{n}(x) \times (E + \Lambda(x)\mathbf{n}(x) \times H) = 0,\ x \in \partial\Omega\}$ for the functional space to look solution wherein.

In "usual" materials, $[\varepsilon]$ and $[\mu]$ are bounded and positively definite so the multiplication operators $K_0(p)$ and $K_1(p)$, in Laplace transform, are bounded and uniformly coercive (with respect to $x \in \Omega$) for all $p = iw + \gamma \in \mathbb{C}$ such that $\gamma > 0$. As the operator $(\mathbb{M}, \mathbf{H})$ is closed maximal dissiptive [8], taking the limit $\gamma \to 0$ leads to the well-posedness of the system (1), (2) for all $p = iw$.

In some examples linked with the metamaterials, the permitivity and/or the permeability, which are functions of the pulsation w, become negative definite for some w thus the well-posedness of the system (1), (2) can't be, a priori, issued. Note that system (1) can both modelize photonic crystals with DBF equations, and electromagnetic phenomenons with Maxwell's equations by setting $K_1 = \mathbb{I}_6$.

But, we have the following result [2]:

Theorem 1. *Let D be a domain of $\mathbb{C}$ such that $D \cap i\mathbb{R} \neq \emptyset$ and let's assume:*

- *$(H1)$: $K_j(p, .) \in L^\infty(\Omega)$, $j = 0, 1$ and $K_1(p, x)$ is invertible $\forall(p, x) \in D \times \Omega$,*
- *$(H2)$: $K_j(., x)$ is holomorphic for $p \in D$.*
- *$(H3)$: There exists $p_0 \in D$ such that $K_j(p_0)$ for $j = 0, 1$ is coercive.*

Then, under assumptions $(H1)-(H2)-(H3)$, the problem (1) is well-posed with compact resolvant for all p in D except for a discrete set of values in D.

Theorem 1 gives an existence and uniqueness result for the generalized DBF system (1). $(H1)-(H2)$ are the descriptive and phenomenological hypotheses which describe the considered media. They ask for piecewise bounded indexes depending smoothly on the Laplace variable p. Note that meromorphic $K_j(p)$ can be taken into account by removing their poles from D. Hence, for example, Debye's and Lorentz's materials satisfy $(H1)-(H2)$. The hypothesis $(H3)$ have been stated in [11] and can be checked experimentally (see Figure 1 in [7]). It means that the considered media behaves as an "usual" one for some p_0. Then, for this p_0, the generalized DBF system is well-posed. And finally, theorem 1 extends the well-posedness of this system to almost all p in the domain of study D.

3. STUDY OF A PERIODIC ARRAY OF SPLIT-RING-RESONATORS

The Split-Ring-Resonators (SRR) have been introduced by J. B. Pendry in 2000 suggesting to make a perfect lens using negative index materials [4]. Some studies dealing with homogenization using a periodic array of SRR follow [6, 9] but the well-posedness of this system remains unanswered at the best of our knowledge.

The effective parameters of a periodical array of interspaced conducting nonmagnetic split-ring-resonators and continuous wires calculated in [9] has the following expressions:

$$[\varepsilon(p, x)] = \left(1 + \frac{w_p^2}{p^2}\right)\mathbb{I}_3,\ [\mu(p, x)] = \left(1 + \frac{\delta p^2}{-p^2 - w_0^2 + p\Gamma}\right)\mathbb{I}_3,\ [\beta(p, x)] = 0, \tag{3}$$

where $w_p > 0$ is the plasma pulsation of gold, $w_0 = \sqrt{\frac{3l}{\pi^2\mu_0 C r^3}}$ and $\Gamma = \frac{2l\rho}{r\mu_0} > 0$ are constants. Here ρ is the resistance per unit length of the rings measured around the circumference, l is the distance between layers, a is the lattice parameter, r is defined on Figure 1 in [9] and C is the capacitance associated with the gaps between the rings.

First, remark that $[\varepsilon]$ and $[\mu]$ defined by (3) are free from x. Futhermore, they depend on p as rational fractions thus $K_0(p)$ and $K_1(p)$ defined by (2), (3) are holomorphic on $D = \mathbb{C}\backslash Z$ where Z is the set of zeros of their denominators. Namely $Z = \left\{\frac{\Gamma-\sqrt{\Gamma^2-4w_0^2}}{2}, 0, \frac{\Gamma+\sqrt{\Gamma^2-4w_0^2}}{2}\right\}$ where $\sqrt{y}$ is equal to $i\sqrt{-y}$ if $y < 0$. Hypothese $(H1) - (H2)$ are then satisfied on $D \times \Omega$. Now, let

$$f : p \in \mathbb{R}\backslash\{0\} \longrightarrow -p^2 - w_0^2 + p\Gamma \in \mathbb{R}.$$

The maximum of f is reached for $p_0 = \frac{\Gamma}{2}$. As $f(p_0) > 0$, $K_0(p_0)$ and $K_1(p_0)$ defined by (2), (3) are coercive thus $(H3)$ is verified. Hence, homogenized parameters $[\varepsilon]$, $[\mu]$ in (3) define an admissible problem in the sence of theorem 1 which can be lately mathematically and numerically investigated.

4. STUDY OF A HOMOGENIZED DIELECTRIC-CHIRAL PHOTONIC CRYSTALS

Let ε_c, μ_c, β_c be the parameters of a chiral media. The photonic crystal, hosted in the vacuum, formed by a periodic array of chiral spheres can be described as a homogenised chiral medium (noted Ω) whose coefficients can be written as follow [7]:

$$\begin{aligned}
[\varepsilon(p,x)] &= \frac{\{(2+\varepsilon_c-2\delta(1-\varepsilon_c))\}\{2+\mu_c-2\delta(1-\mu_c)\}+4(\delta-1)^2\varepsilon_c\mu_c p^2\beta_c^2}{\{(2+\varepsilon_c+\delta(1-\varepsilon_c))\}\{2+\mu_c-2\delta(1-\mu_c)\}-2(\delta-1)(\delta+2)\varepsilon_c\mu_c p^2\beta_c^2}\mathbb{I}_3,\\
[\mu(p,x)] &= \frac{\{(2+\varepsilon_c-2\delta(1-\varepsilon_c))\}\{2+\mu_c-2\delta(1-\mu_c)\}+4(\delta-1)^2\varepsilon_c\mu_c p^2\beta_c^2}{\{(2+\varepsilon_c-2\delta(1-\varepsilon_c))\}\{2+\mu_c+\delta(1-\mu_c)\}-2(\delta-1)(\delta+2)\varepsilon_c\mu_c p^2\beta_c^2}\mathbb{I}_3,\\
[\beta(p,x)] &= \frac{9\delta\varepsilon_c\mu_c\beta_c}{\{(2+\varepsilon_c-2\delta(1-\varepsilon_c))\}\{2+\mu_c-2\delta(1-\mu_c)\}+4(\delta-1)^2\varepsilon_c\mu_c p^2\beta_c^2}\mathbb{I}_3,
\end{aligned} \tag{4}$$

where $0 < \delta < 1$ is the fraction of the total volume occupied by the spheres.

Notice first that $[\varepsilon]$, $[\mu]$ and $[\beta]$ defined by (4) are free from x. Thus coefficients K_0 and K_1 defined by (2)–(4) are bounded with respect to $x \in \Omega$. Futhermore $[\varepsilon(p)]$, $[\mu(p)]$, $[\mu(p)][\beta(p)]$ and $[\varepsilon(p,x)][\beta(p,x)]$ depend on p as rational fraction. Thus the holomorphy of K_0 and K_1 holds on $D = \mathbb{C}\backslash Z$ where Z is the set of zeros of their denominators. This implies that:

$$Z=\left\{\pm\sqrt{\frac{\{2+\varepsilon_c+\delta(1-\varepsilon_c)\}\{2+\mu_c-2\delta(1-\mu_c)\}}{2(\delta-1)(\delta+2)\varepsilon_c\mu_c\beta_c^2}},\pm\sqrt{\frac{\{2+\varepsilon_c-2\delta(1-\varepsilon_c)\}\{2+\mu_c+\delta(1-\mu_c)\}}{2(\delta-1)(\delta+2)\varepsilon_c\mu_c\beta_c^2}}\right\}, \tag{5}$$

with the same notation of $\sqrt{.}$ as previously. So hypotheses $(H1) - (H2)$ are satisfied on $D \times \Omega$ where $D = \mathbb{C}\backslash Z$. We remark that $K_0(1)$ and $K_1(1)$ defined by (2)–(4) are coercive. Hence, the physical modelling which gives the parameters defined by (4) leads to an admissible DBF system in the sence of theorem 1. This problem can be lately numerically and mathematically studied.

Remark 1. *The fraction of the total volume occupied by the spheres, noted δ, can depend on $x \in \Omega$. So $[\varepsilon]$, $[\mu]$ and $[\beta]$ defined in (4) will be able to depend on x too. The study of the well-posedness of the system (1), (2), (4) with $\delta = \delta(x)$ is not slighty different from the previous case. The main modification occurs on the definition of the set Z (5), now noted Z': $p \in Z' \iff \forall x \in \Omega$,*

$$\left\{\begin{array}{l}\{2+\varepsilon_c-2\delta(x)(1-\varepsilon_c)\}\{2+\mu_c+\delta(x)(1-\mu_c)\}-2(\delta(x)-1)(\delta(x)+2)\varepsilon_c\mu_c p^2\beta_c^2=0,\\ \{2+\varepsilon_c+\delta(x)(1-\varepsilon_c)\}\{2+\mu_c-2\delta(x)(1-\mu_c)\}-2(\delta(x)-1)(\delta(x)+2)\varepsilon_c\mu_c p^2\beta_c^2=0.\end{array}\right.$$

Remark that Z' is the disjoint union of four intervals located on the imaginary axis. Moreover, $Z' \cap \{0\} = \emptyset$ and, if δ is free from x, $Z = Z'$. As for all x in Ω, $0 < \delta(x) < 1$, hypotheses $(H1) - (H2)$ are satisfied on $D \times \Omega$ where $D = \mathbb{C}\backslash Z'$. $K_0(1)$ and $K_1(1)$ still define coercive multiplication operators. Thus $(H3)$ is checked. Hence the same conclusion as previously (with δ free from x) holds.

5. CLOAKING AND BACK-SCATTERING INVISIBILITY WITH PML

At last we study some absorbing boundary conditions of PML type. Using the result of the Section 2, we show that Maxwell's equations with convex PML are well-posed. We also show theoretically that PML can perform cloaking and back-scattering invisibility. Moreover we give some numerical results showing these properties using finite volumes method for T. M. Maxwell's equations.

Let $x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ and consider the following T.M Maxwell's system:

$$\left\{ B(p)\begin{pmatrix} e_1\\ e_2\\ h_3\end{pmatrix} + \begin{pmatrix} 0 & 0 & -\partial_y\\ 0 & 0 & \partial_x\\ -\partial_y & \partial_x & 0\end{pmatrix}\begin{pmatrix} e_1\\ e_2\\ h_3\end{pmatrix} = f, \right. \tag{6}$$

where

$$B(p) = \begin{pmatrix} \frac{\widetilde{\gamma}(p,x)}{\gamma(p,x)} & 0 & 0\\ 0 & \frac{\gamma(p,x)}{\widetilde{\gamma}(p,x)} & 0\\ 0 & 0 & \gamma(p,x)\widetilde{\gamma}(p,x)\end{pmatrix}, \tag{7}$$

acts as absorbing boundary condition of convex PML type on a subset Θ of Ω. Two different types of PML will be used according to the geometry studied: circular PML and cartesian PML.

Circular PML are defined on an annulus $\Gamma \subset \Omega$ of radius r_1 and r_2 by (7) with:

$$\left\{ \gamma(p,x) = 1 - \frac{\sigma_\rho(\rho)}{p},\ \widetilde{\gamma}(p,x) = 1 - \frac{\int_{\rho_1}^{\rho} \sigma_\rho(s)ds}{p\rho}, \right. \tag{8}$$

where $\rho = |x|$, $\sigma_\rho(\rho) \geq 0$ for x in Γ and $\sigma_\rho(\rho) = 0$ for x in $\Omega\backslash\overline{\Gamma}$.

Cartesian PML are defined on $]-a,+a[\times]-b,+b[\subset \Omega$, by (7) with:

$$\left\{ \widetilde{\gamma}(p,x) = \frac{\sigma_2(x_2)+p}{p},\ \gamma(p,x) = \frac{\sigma_1(x_1)+p}{p}, \right. \tag{9}$$

where $\sigma_{x_1}(x_1) = 0$ for x_1 in $]-a,+a[$ and $\sigma_{x_1}(x_1) > 0$ for x_1 in $\mathbb{R}\backslash]-a,+a[$, $\sigma_{x_2}(x_2) = 0$ for x_2 in $]-b,+b[$ and $\sigma_{x_2}(x_2) > 0$ for x in $\mathbb{R}\backslash]-b,+b[$.

We use Finite Volume method for T.M Maxwell's equation (6), (7) to investigate numerically the cloaking and back-scattering invisibility properties. The cloaking effect is numerically shown on an annulus with circular PML (8). The back-scattering abilities has been computed with cartesian PML (9) on a square. A point source is introduced. It is described in Equation (6) with $f = \widetilde{f}[0,0,1]^T$ where $\widetilde{f}$ is a mollification of δ_{x_0} supported by the open disk of radius $\eta = \frac{\pi}{1.4}$:

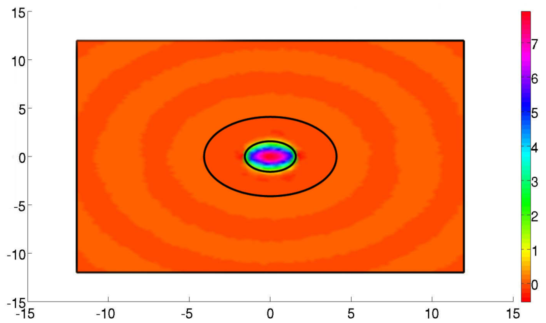
$$\widetilde{f}(x) = \left\{ \begin{array}{l} 0, |x - x_0| \geq \eta, \\ \exp(\eta)\exp\left(\frac{-\eta}{|x-x_0|^2}\right), |x - x_0| < \eta. \end{array} \right. \tag{10}$$

Figure 1(a) represents the approximation of the field $\mathrm{Re}(h_3)$ solution of T. M. Maxwell's equations (6)–(8), (10) with $x_0 = (0,0)$ and $\sigma_\rho(\rho) = 5,617\exp(-(\rho - \frac{r_1+r_2}{2})^2)$. We can see that PML absorbs strongly the field $\mathrm{Re}(h_3)$. Moreover it is reflectionless thus electromagnetic cloaking can be performed with PML. The field outside the annulus, the concentric circles, is closed to zero ($\approx \pm 0.2$). Inside the annulus, we have exactly the restriction of the solution of (6)–(8), (10) propagating on unbounded space.

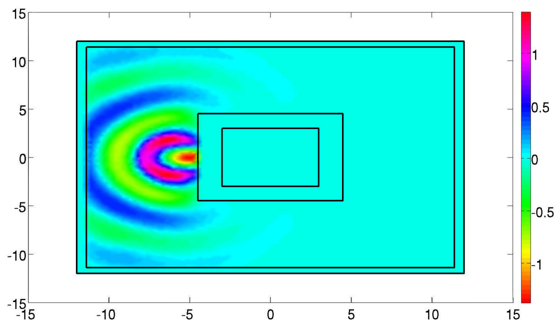
Figure 1(b) represents the approximation of the field $\mathrm{Re}(h_3)$ solution of T. M. Maxwell's equations (6), (7), (10), (9) with $x_0 = (-5,0)$, $\sigma_j^k(x_j) = (x_j \pm d_k)^2$. There are two layers of PML. One on the boundary of our computational domain, for numerical purpose, defined by (9) with $\sigma^1(x_j) = (x_j \pm 12)^2$. The other, which is lying in the middle of the aera, for invisibility aim, is (9) with $\sigma^2(x_j) = (x_j \pm 3)^2$. We can see that a measurement of the field at $x = (x_1^m, x_2^m)$ with $x_1^m \leq -5$ and $x_2^m = 0$ is not modified by the presence of the internal PML box. Hence back scattering invisibility is performed.

The electromagnetic wave propagation in vacuum is approached by the system (6), (7). The matrix $B(p,x)$ is bounded and holomophic on $(\mathbb{C}\backslash\{0\}) \times \Omega$ thus satisfy $(H1) - (H2)$. As the multiplication operator $B(1)$ is coercive, $(H3)$ is checked with $p_0 = 1$. Again, the parameters in (7), (8) and (7), (9) define an admissible system in the sence of theorem 1.

This result and the numerical simulations (see Figures 1(a) and (b)) lead us to investigate some properties of the PML. Even if PML are not by nature, we investigate their properties to propose



(a)



(b)

Figure 1: (a) Cloaking. (b) Back-scattering invisibility.

a distribution of indexes performing cloaking and back-scattering invisibility. It is well known that PML is reflectionless and absorbs strongly the scattered waves. Then, PML can performs electromagnetic cloaking. To show that PML can perform back scattering invisibility, we assume, for simplicity, that $\Theta \subset \Omega$ is a circle. We use the following complex formulation of PML [5]:

$$\widetilde{x} : \mathbb{R}^3 \longrightarrow \mathbb{C}^3, x \longrightarrow \widetilde{x}(x) = \widetilde{\rho}(\cos(\theta), \sin(\theta)),$$

where $\widetilde{\rho_x} = \rho_x + \frac{K(\rho)}{p}$, $\theta \in [0, 2\pi]$ and $K \geq 0$.

Let $y \in \mathbb{R}^2$, from [5], the solution of the Maxwell's equations (6), (7) with $f(x) = \delta_y(x)$ is a radial function $\widetilde{G}(|\widetilde{x} - \widetilde{y}|)$. Let $f \in L^2(\Omega)$ be compactly supported in Ω. The solution of (6), (7) is given by $\widetilde{u}(x) = \int_\Omega \widetilde{G}(|\widetilde{x}(x) - \widetilde{y}(y)|) f(\widetilde{y}(y)) dy$. Let $G(|x - y|)$ be the Green's function of the T. M. Maxwell's equations (6) in the vacuum ($B(p) = \mathbb{I}_3$) thus the solution of (6) is given by $u(x) = \int_\Omega G(|x - y|) f(y) dy$. A straightforward calculation of $|\widetilde{x} - \widetilde{y}|$ leads to:

$$|\widetilde{x} - \widetilde{y}|^2 = \widetilde{\rho_x}^2 + \widetilde{\rho_y}^2 - 2\{\cos(\theta_x)\cos(\theta_y) + \sin(\theta_x)\sin(\theta_y)\}.$$

Thus $|\widetilde{x} - \widetilde{y}|^2 = \widetilde{\rho_x}^2 + \widetilde{\rho_y}^2 - 2\{\cos(\theta_x - \theta_y)\}$, and, developing $\widetilde{\rho_x}$, $\widetilde{\rho_y}$, and assuming $\theta_x = \theta_y[2\pi]$, it follows:

$$|\widetilde{x} - \widetilde{y}|^2 = (\rho_x - \rho_y)^2.$$

Then, for all x such that $\theta_x = \theta_y[2\pi]$, we have $\widetilde{u}(x) = u(x)$ so the solution of (6), (7) and the solution of (6) in the vaccum match and we can conclude that PML performs back-scattering invisibility.

Remark that the proof of back-scattering invisibility performed by PML can be done in $\mathbb{R}^3$ with spherical PML for Maxwell's equations.

6. CONCLUSION

In this paper we were interested in analyzing scattering by complex materials either for Drude-Born-Fedorov system or Maxwell's equations. Using a general mathematical framework, we studied a Maxwell's equations with a periodical array of interspaced Split-Ring-Resonators. A DBF system in presence of homogenized dielectric-chiral photonic crystals have been studied. The Maxwell's system with some absorbing boundary condition of PML type, seen as "ideal" metamaterial, have also been considered. At last, we checked both numerically (using Finite Volume method for T.M Maxwell's system) and theoretically the cloaking and back-scattering invisibility abilities of the PML. The homogenized systems treated in this paper have been analyzed with a general mathematical framework. Futhermore, this tool works under compatible assumptions relevant for the physical modelling thus more homogenized systems can be studied using it.

It remains to find how we can approximate the solution of these homogenized systems. The classical numerical methods (Finite Volumes, Finite Elements, Finite Difference, ...) used for the approximation of Maxwell's equations or DBF system in presence of "usual" materials are convergent mainly due to the positiveness of the parameters $[\varepsilon]$ and $[\mu]$. The question is then to know if these numerical schemes are still convergent in presence of materials whose parameters can become negative definite for some pulsation w. An answer can be found in some works dealing with finite element methods for wave propagation in metamaterials which have been made in [1, 10] modifying the Finite Element scheme. However, these result can't be, at the best of our knowledge, straightforwardly used for systems like Maxwell's equations with a periodical arrangement of split-ring-resonators or for the DBF system with chiral photonic crystals.

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